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The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms

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Abstract: The vacuum structure and spectra of two-dimensional gauge theories with $\mathcal{N} = (2, 2)$ supersymmetry are investigated. These theories admit a twisted mass term for charged chiral matter multiplets. In the case of a $U(1)$ gauge theory with N chiral multiplets of equal charge, an exact description of the BPS spectrum is obtained for all values of the twisted masses. The BPS spectrum has two dual descriptions which apply in the Higgs and Coulomb phases of the theory respectively. The two descriptions are related by a massive analog of mirror symmetry: the exact mass formula which is given by a one-loop calculation in the Coulomb phase gives predictions for an infinite series of instanton corrections in the Higgs phase. The theory is shown to exhibit many phenomena which are usually associated with $\mathcal{N} = 2$ theories in four dimensions. These include BPS-saturated dyons which carry both topological and Noether charges, non-trivial monodromies of the spectrum in the complex parameter space, curves of marginal stability on which BPS states can decay and strongly coupled vacua with massless solitons and dyons.

Keywords: Field Theories in Lower Dimensions, Solitons Monopoles and Instantons, Supersymmetry and Duality.

JHEP11(1998)005

Contents

1. Introduction

The purpose of this paper is to present some exact results for the mass spectrum of abelian gauge theories in two dimensions with $\mathcal{N} = (2, 2)$ supersymmetry. In recent work [1], Hanany and Hori introduced a new relevant parameter for these theories: a twisted mass for chiral superfields which corresponds to the expectation value of a background twisted chiral multiplet. [1] The main new result presented below is an exact description of the spectrum of BPS states as a function of the twisted masses. Although most of the results presented here can easily be generalized to other $\mathcal{N} = (2, 2)$ theories, I will consider a model with gauge group $U(1)_G$ and N chiral multiplets of equal

[1]Details of the multiplets of $\mathcal{N} = (2, 2)$ supersymmetry in two dimensions and the corresponding superfields are reviewed in section 2 below.

charge. Without twisted masses this theory reduces to the supersymmetric CP^{N-1} σ-model at low energy and its exact spectrum is well known [2, 3, 4]. In contrast, I will argue that the theory with non-zero twisted masses exhibits many phenomena which are new in two dimensions but familiar in the context of $\mathcal{N} = 2$ theories in four dimensions [5, 6]. These include BPS dyons which carry both topological and Noether charges, a two-dimensional analog of the Witten effect [7], non-trivial monodromies of the spectrum, curves of marginal stability on which BPS states can decay and strong-coupling vacua with massless solitons and dyons. The correspondence between two- and four-dimensional theories will be made precise below by identifying a complex curve whose periods govern the BPS spectra of both models.

Two-dimensional $\mathcal{N} = (2, 2)$ gauge theories have been studied extensively in the past because of their close relation to world-sheet conformal field theories which arise in compactifications of Type II string theory on Calabi-Yau manifolds and the phenomenon of mirror symmetry [8, 9]. Even with zero twisted masses, the model considered here is not conformally invariant but has a mass gap generated by strong quantum fluctuations in the infra-red. Nevertheless the model has a property which is a massive analog of mirror symmetry. In two different regions of parameter space, the theory is realized in a Higgs phase where the $U(1)$ gauge symmetry is spontaneously broken and a Coulomb phase with an unbroken gauge symmetry. An exact formula for BPS masses can be derived in the Coulomb phase by a one-loop calculation. The same formula applies in the Higgs phase where it predicts the exact numerical coefficients of an infinite series of instanton corrections. The main evidence in favour of the results presented here comes from a feature of the theory which has not been utilized before: the Higgs phase description of the theory is weakly coupled for large values of the twisted masses. These results are described in the remainder of this section while further details of the corresponding calculations appear in the subsequent sections. Theories with $\mathcal{N} = (2, 2)$ supersymmetry have recently been discussed from a different point of view in [10, 11].

A gauge multiplet of $\mathcal{N} = (2, 2)$ supersymmetry in two dimensions contains a complex scalar σ which is the lowest component of a twisted chiral superfield Σ. The theory considered here also contains N chiral superfields, Φ_i $i = 1, 2, \ldots, N$, each of charge $+1$ under $U(1)_G$, whose lowest components are complex scalars ϕ_i. The parameters of the theory include a gauge coupling e, which has the dimensions of mass, as well as a Fayet-Iliopoulis (FI) parameter, r, and vacuum angle, θ. The FI parameter and vacuum angle are dimensionless and it is convenient to combine them as a single complex parameter, $\tau = ir + \theta/2\pi$. As mentioned above, it is also possible to include a twisted mass m_i for each chiral superfield Φ_i. Only the differences between the twisted masses are physically significant: $\sum_{i=1}^{N} m_i$ can be set to zero by a linear shift in σ. In the absence of central charges, the $\mathcal{N} = (2, 2)$ supersymmetry algebra has two abelian R-symmetries denoted $U(1)_R$ and $U(1)_A$. More generally, the supersymmetry algebra can be modified by including a central charge which breaks one of the two R-symmetries. In this case the spectrum of the corresponding theory can include massive BPS saturated

JHEP11(1998)005

states which lie in special, short representations of supersymmetry [12, 13]. In the following, the vacuum structure and the spectrum BPS states in the classical theory and in the corresponding quantum theory will be considered in turn.

1.1. The classical theory

The classical theory with zero twisted masses has the maximal R-symmetry group, $U(1)_R \times U(1)_A$. The massless theory also has an $SU(N)$ global symmetry under which the chiral multiplets transform in the fundamental representation. For $r = 0$, the theory has a classical Coulomb branch, with $\phi_i = 0$ and σ unconstrained. In these vacua the $U(1)_G$ gauge symmetry is unbroken and the gauge multiplet fields are classically massless. In contrast, for $r > 0$, the D-term conditions for a supersymmetric vacuum set $\sigma = 0$ and require that,

$$\sum_{i=1}^{N} |\phi_i|^2 = r\,. \tag{1.1}$$

The resulting space of gauge-inequivalent classical vacua is a copy of CP^{N-1}. At each point on this vacuum manifold at least one of the charged scalars, ϕ_i, is non-zero and the $U(1)_G$ gauge-symmetry is spontaneously broken. This is the classical Higgs branch of the theory. The theory on the classical Higgs branch has $N-1$ massless chiral multiplets corresponding to the flat directions tangent to the vacuum manifold. The remaining degrees of freedom get masses of order $\sqrt{r}e$ by the Higgs mechanism. The effective theory for energies much less than this mass scale is a supersymmetric σ-model with target space CP^{N-1}. The coupling constant of the σ-model is related to the FI parameter of the underlying gauge theory as $g \sim 1/\sqrt{r}$. The CP^{N-1} target space is covered by N different coordinate patches, $\mathcal{P}_j$ with $j = 0, 1, \ldots, N$, each with $N-1$ complex coordinates $w_i^{(j)} = \phi_i/\phi_j$ with $i \neq j$. The patch $\mathcal{P}_j$ covers the complement in CP^{N-1} of the submanifold defined by $\phi_j = 0$.

The inclusion of twisted masses, with $m_i \neq m_j$ for each i and j, changes the classical analysis given above in several ways. First, as shown in section 3 below, the continuous vacuum degeneracy discovered above is lifted and the model has N isolated supersymmetric vacua. Specifically there is exactly one vacuum $\mathcal{V}_i$ in each of the coordinate patches $\mathcal{P}_i$ defined above. In each vacuum $\mathcal{V}_i$ the low-energy effective theory is a variant of the supersymmetric CP^{N-1} σ-model with explicit mass terms for each of the $N-1$ gauge-invariant fields $w_j^{(i)} = \phi_j/\phi_i$ with $j \neq i$ (and their superpartners). Second, the $SU(N)$ global symmetry of the massless theory is explicitly broken to its maximal abelian subgroup $U(1)^{N-1} = (\otimes_{i=1}^{N} U(1)_i)/U(1)_G$. Here $U(1)_i$ denotes the global symmetry with generator S_i under which ϕ_j has charge $+1$ if $j = i$ and zero otherwise. Finally, in the theory with non-zero twisted masses, the R-symmetry $U(1)_A$ is broken down to a discrete subgroup, Z_2. As mentioned above this permits a non-zero central charge to appear in the supersymmetry algebra, which is a necessary condition

JHEP11(1998)005

for the existence of BPS states. This possibility is analysed in detail in section 4. It is shown that the classical spectrum includes three different kinds of BPS states,

1) In the vacuum $\mathcal{V}_i$, the BPS spectrum includes the elementary quanta of the $N-1$ σ-model fields $w_j^{(i)}$ with $j \neq i$ (and superpartners). These states carry the global $U(1)$ Noether charges $\vec{S} = (S_1, S_2, \ldots S_N)$. Including the states from each vacuum the spectrum includes BPS states with all $N(N-1)$ possible charge vectors of the form $\pm(0, \ldots, +1, \ldots, -1, \ldots, 0)$.

2) A two-dimensional theory with isolated vacua can have topologically stable solitons. These are classical field configurations which asymptote to one vacuum, $\mathcal{V}_L$, at left spatial infinity and a different vacuum, $\mathcal{V}_R$, at right spatial infinity. In this connection it is useful to define topological charges T_i which are equal to $+1$ if $\mathcal{V}_R = \mathcal{V}_i$, -1 if is $\mathcal{V}_L = \mathcal{V}_i$, and zero otherwise. The theory with twisted masses has Bogomol'nyi saturated solitons which interpolate between each pair of vacua, $\mathcal{V}_i$ and $\mathcal{V}_j$ with $i \neq j$. The solitons yield $N(N-1)$ BPS multiplets of $\mathcal{N} = (2,2)$ supersymmetry. These states carry the topological charges $\vec{T} = (T_1, T_2, \ldots T_N)$. As for the Noether charges, the spectrum includes all states with charge vectors of the form $\pm(0, \ldots, +1, \ldots, -1, \ldots, 0)$.

3) The solitons of the model have a feature which is unusual for topologically stable kinks in two dimensions: they have an internal degree of freedom corresponding to global $U(1)$ rotations. Specifically, the soliton which interpolates between the vacua $\mathcal{V}_i$ and $\mathcal{V}_j$ transforms under the global $U(1)$ symmetry with generator $S_j - S_i$. Quantizing this degree of freedom yields an infinite tower of 'dyons' which carry both Noether and topological charges. For each allowed topological charge vector $\vec{T}$, the spectrum contains BPS dyons with Noether charge vectors $\vec{S} = S\vec{T}$ where S can be any integer. These dyons exhibit an exact analog of the Witten effect [7] in four-dimensions; in the presence of a non-zero vacuum angle their global $U(1)$ charge is shifted by an amount $\theta/2\pi$. Similar two-dimensional dyons have been studied previously in [15].

The masses of all the states described above are given by a BPS mass formula $M = |Z|$ with

$$Z = -i\vec{m} \cdot (\vec{S} + \tau\vec{T}) \,, \tag{1.2}$$

where $\vec{m} = (m_1, m_2, \ldots, m_N)$. The classical BPS spectrum described above coincides exactly with the classical spectrum of massive BPS states of $\mathcal{N} = 2$ supersymmetric $SU(N)$ Yang-Mills theory in four dimensions. In particular, in the spectrum of the two-dimensional theory there is a one-to-one correspondence between elementary particles and solitons of exactly the same form as that noted by Goddard, Nuyts and Olive [16] for the classical $D = 4$ gauge theory. The global $SU(N)$ symmetry of the two-dimensional theory corresponds to the gauge symmetry of the four-dimensional theory. The two-dimensional twisted masses m_i correspond to the eigenvalues of the adjoint scalar VEV which breaks $SU(N)$ down to $U(1)^{N-1}$ on the Coulomb branch of the four-dimensional

JHEP11(1998)005

theory. Hence the components of $\vec{S}$ correspond to the abelian electric charges in $D = 4$ and those of $\vec{T}$ to the magnetic charges. The complex coupling $\tau = ir + \theta/2\pi$ is mapped onto the complexified gauge coupling $\tau_{4D} = 4\pi i/g_{4D}^2 + \theta_{4D}/2\pi$ of the four-dimensional theory. The correspondence between the couplings and symmetries of the two theories was noted in [1] and explained in the context of intersecting D-branes in type IIA string theory. It would be interesting to try an understand the relation described here between the spectra of the two theories in the same way.

1.2. The quantum theory

The classical theory described above is modified in several ways by quantum effects. The FI parameter runs logarithmically at one-loop and is traded for an RG-invariant scale Λ by dimensional transmutation. In addition, the $U(1)_A$ R-symmetry of the theory without twisted masses is broken by an anomaly to a residual discrete symmetry Z_{2N}. This means that the bare θ-parameter can be set to zero by a chiral rotation of the fields (a phase rotation of the twisted masses is also necessary if they are non-zero). The physical parameters of the quantum theory are the gauge coupling e, the dynamical scale Λ and the twisted masses m_i. The are two different regions of this parameter space in which the theory has a weakly-coupled description and its properties can be analysed reliably.

a) $e >> |m_i - m_j| >> \Lambda$**.** In this regime the low-energy theory is described by the classically massive version of the supersymmetric CP^{N-1} σ-model discussed in the previous section. Although the σ-model is asymptotically free its coupling only runs for energy scales larger than the masses $|m_i - m_j|$. Hence the low-energy theory is weakly-coupled as long as $|m_i - m_j| >> \Lambda$. In this region of parameter space the $U(1)_G$ gauge symmetry is spontaneously broken and the BPS spectrum is qualitatively similar to that of the classical theory described above. The classical spectrum is corrected by one-loop effects which are calculated explicitly for the $N = 2$ case in section 5. There are also non-perturbative corrections from all numbers of σ-model instantons.

b) $e << \Lambda$**.** The theory in this regime consists a light gauge multiplet weakly coupled to massive chiral ones [8]. In particular the dimensionful gauge coupling is much smaller then the other relevant mass scales and the model can be analysed using ordinary perturbation theory. Note that this is a completely different expansion to the perturbation in the σ-model coupling considered above above. A one-loop calculation suffices to show that the theory has N isolated vacua, each with unbroken $U(1)_G$ gauge symmetry [8]. The BPS spectrum consists entirely of solitons which are charged under $U(1)_G$ and interpolate between different vacua. In the absence of twisted masses the solitons lie in multiplets of the unbroken $SU(N)$ global symmetry. Introducing small twisted masses breaks this symmetry and introduces mass splittings between degenerate states.

Superficially, it appears that the descriptions of the theory in these two regions of parameter space are completely different. For $e >> \Lambda$, the gauge symmetry is spontaneously broken and the theory is in a Higgs phase. In contrast, for $e << \Lambda$, $U(1)_G$ is unbroken and, adopting the terminology of four dimensional gauge theories, the theory is in its Coulomb phase.[2] Despite these differences, some features of the theory remain the same in both phases. The simplest example is the Witten index which counts the number of supersymmetric vacua weighted by fermion number. Two-dimensional theories with $\mathcal{N} = (2, 2)$ supersymmetry also have another supersymmetric index which was introduced by Cecotti, Fendley, Intriligator and Vafa (CFIV) in [17]. This index is invariant under D-term variations of the superspace Lagrangian. This is a refinement of the Witten index as the latter is invariant under both F- and D-term variations (subject to certain boundary conditions). While only the vacuum states of the theory contribute to the Witten index, the CFIV index is sensitive to all states in short representations of supersymmetry. In fact, part of the information contained in the index is the mass and degeneracy of each BPS saturated state [4].

The CFIV index is relevant in the present context because the gauge kinetic term can be written as a D-term in $\mathcal{N} = (2, 2)$ superspace [8]. In addition, the fields can be rescaled so that the gauge coupling only appears in this term in the action. It follows from the above discussion that the masses of BPS states are independent of e. For this reason we can calculate BPS masses using the weakly-coupled Coulomb phase description of the theory which is valid for $e << \Lambda$ and apply the results for all values of e. This calculation is described in section 6. The result is that, for all values of the parameters, the mass of a BPS state with global Noether charge $\vec{S}$ and topological charge $\vec{T}$ is given by $M = |Z|$ with

$$Z = -i(\vec{m} \cdot \vec{S} + \vec{m}_D \cdot \vec{T})\,, \tag{1.3}$$

where $\vec{m}_D = (m_{D1}, m_{D2}, \ldots, m_{DN})$ and $m_{Di} = Ne_i - \sum_{j=1}^{N} m_j \log(e_i + m_j)$. Here e_i, with $i = 1, 2, \ldots, N$ denote the roots of the polynomial equation

$$\prod_{i=1}^{N}(x + m_i) - \tilde{\Lambda}^N = 0\,, \tag{1.4}$$

where $\tilde{\Lambda} = \Lambda \exp(-1 + i\theta/N)/2$. For $|m_i - m_j| >> \tilde{\Lambda}$, (1.3) can be compared directly with semiclassical results obtained using the massive σ-model description of the theory. The exact soliton and dyon masses have a non-trivial expansion in the small parameters $\Lambda/|m_i - m_j|$ which corresponds to the weak-coupling expansion of the σ-model. In general the expansion contains terms corresponding to one-loop perturbation theory as well as an infinite series of corrections which can be interpreted as coming from

[2] In two dimensions a Coulomb interaction between two charges leads to a confining linear potential. However in the supersymmetric theory considered here the gauge multiplet gets a mass from quantum effects and the Coulomb interaction is screened [13, 14]. Thus there are no long-range gauge interactions in either phase.

JHEP11(1998)005

σ-model instantons. In the simplest case $N = 2$, the exact formula predicts a one-loop correction to the classical spectrum (1.2) which is equivalent to the replacement $\tau \to \tau_{\text{eff}} + 1$ where $\tau_{\text{eff}} = i \log(m/\tilde{\Lambda})/\pi$ and $m = m_1 - m_2$. In section 5 this result is tested against an explicit semiclassical calculation of quantum corrections to the soliton mass. The above results also predict two-dimensional analogs for several phenomena which occur in four-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry:

1) The branch-cuts appear in the exact formula for $\vec{m}_D$ implies that the BPS spectrum undergoes monodromies in the complex parameter space. In the $N = 2$ case, with $m = m_1 - m_2$, there is a non-trivial monodromy around the point at infinity in the complex m-plane. In this case, the charge vectors $\vec{S}$ and $\vec{T}$ transform as

$$\vec{S} \to \vec{S} - 2\vec{T}\,, \qquad \vec{T} \to \vec{T}\,. \tag{1.5}$$

In sections 5 this effect is derived explicitly using weak-coupling methods.

2) There are submanifolds of the parameter space on which the roots of equation (1.4) become degenerate. On these submanifolds BPS states become massless. In particular, solitons and dyons which are very massive at weak coupling can become massless at strong coupling for some values of the parameters.

3) There can be curves of marginal stability (CMS) on which BPS states can decay. Typically these curves will have real codimension one in the parameter space and therefore can disconnect different regions of this space. These curves play an important role in resolving the remaining discrepancies between the BPS spectra in different regions of parameter space. For $|m_i - m_j| >> \Lambda$ the semiclassical spectrum described above includes an infinite number of stable BPS states. In contrast, for $|m_i - m_j| << \Lambda$, the BPS spectrum should be close to that of the supersymmetric CP^{N-1} σ-model which has only a finite number of such states. This disparity can be resolved if the regions of large and small twisted mass are separated by a curve of marginal stability. Some explicit checks that the required CMS is present in the $N = 2$ case are performed in section 6.

These effects suggest that there is a correspondence between the two-dimensional theory with twisted masses and an $\mathcal{N} = 2$ supersymmetric gauge theory in four-dimensions which holds at the quantum level. This can be made precise by noting that the soliton masses implied by the exact formula (1.3) correspond to the periods of the following degenerate elliptic curve;

$$(t - \tilde{\Lambda}^N)\left(t - \prod_{i=1}^{N}(x + m_i)\right) = 0\,. \tag{1.6}$$

This is the same curve which describes an $\mathcal{N} = 2$ gauge theory in four-dimensions with gauge-group $SU(N)$ and N hypermultiplets in the fundamental representation. Specifically, the four-dimensional theory is at a particular point on its Coulomb branch which is the root of the baryonic Higgs branch [18]. One application of this correspondence is that, at least for the $N = 2$ case, the existence of the CMS described in **3** above can be deduced from the known behaviour of the four-dimensional theory [19, 20]. The condition (1.4) for a supersymmetric vacuum in the two-dimensional Coulomb phase can be interpreted as the conditions obeyed by the singular points on the complex manifold (1.6). This phenomenon seems to be a massive generalization of the description of mirror symmetry between $\mathcal{N} = (2, 2)$ conformal theories given in [9].

The results described above leave several questions which require further investigation. In particular it would be very interesting to uncover the physics associated with the singular points where solitons become massless. Given the usual problems with interacting massless particles in two dimensions [21], the IR effective theory at these points may simply be a free theory. Another interesting possibility which may arise is that of a non-trivial conformal field theory. Finally, the origin of the correspondence uncovered here between the BPS spectra of two seemingly very different theories in different spacetime dimensions still requires an explanation. An obvious place to look for such an explanation is the brane construction of [1]. Specifically, configurations of intersecting D2, D4 and NS5 branes in type IIA string theory, which become M2 and M5 branes in M-theory, provide a natural way to relate world-volume gauge theories in two- and four-dimensions.

2. Fields and symmetries and dimensional reduction

This section contains a review of the basic features of theories with $\mathcal{N} = (2, 2)$ supersymmetry in two dimensions. These theories were studied in detail by Witten in [8] and the presentation given here closely follows this reference. With a few exceptions, the notation and conventions adopted below are those of [8]. This section also includes a review of BPS saturated solitons in two dimensions.

Theories with $\mathcal{N} = (2, 2)$ supersymmetry in two dimensions (2D) may be obtained by dimensional reduction of four-dimensional theories with $\mathcal{N} = 1$ supersymmetry. Specifically, we will start in four-dimensional Minkowski space with coordinates X_m $m = 0, 1, 2, 3$ and obtain a two-dimensional theory by considering field configurations which are independent of X_1 and X_2. The two-dimensional spacetime coordinate is denoted x_μ with $\mu = 0, 1$ where $x_0 = X_0$ and $x_1 = X_3$. A four-vector A_m reduces to a two-vector a_μ and two real scalars. A left-handed Weyl spinor ψ_α in four-dimensions yields a complex spinor in two dimensions with components $(\psi_-, \psi_+) = (\psi_1, \psi_2)$. Similarly, a right-handed Weyl spinor $\bar{\psi}_{\dot{\alpha}}$ in four-dimensions yields a complex spinor in two dimensions with components $(\bar{\psi}_-, \bar{\psi}_+) = (\bar{\psi}_{\dot{1}}, \bar{\psi}_{\dot{2}})$. The components of ψ_α and $\bar{\psi}_{\dot{\alpha}}$ can be combined to make a two-dimensional Dirac spinor Ψ and its charge conjugate $\bar{\Psi}$. However, following the notation of [8] we will mostly work in terms of the components $\psi_\pm$ and $\bar{\psi}_\pm$ and use the four-dimensional notation for spinors with summation over + and − components for repeated indices.[3] For example we have $\psi^\alpha\psi_\alpha = \psi^-\psi_- + \psi^+\psi_+ = 2\psi_+\psi_-$.

The $\mathcal{N}=1$ superalgebra in four dimensions contains left- and right-handed Weyl supercharges Q_α and $\bar{Q}_{\dot\alpha}$ with anti-commutator,

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^m_{\alpha\dot\alpha} P_m\,. \tag{2.1}$$

On dimensional reduction to two dimensions, we obtain the $\mathcal{N}=(2,2)$ supersymmetry algebra for two Dirac supercharges $Q_\pm$ and $\bar{Q}_\pm$,

$$\begin{array}{ll} \{Q_+, \bar{Q}_+\} = -2(p_0+p_1) & \{Q_-, \bar{Q}_-\} = -2(p_0-p_1) \\ \{Q_-, \bar{Q}_+\} = 2Z & \{Q_+, \bar{Q}_-\} = 2\bar{Z}\,, \end{array} \tag{2.2}$$

where $p_0 = P_0$, $p_1 = P_3$. $p_0 \pm p_1$ correspond to right and left moving momentum in two dimensions. The components of four-momentum in the reduced directions yield a complex central charge $Z = P_1 - iP_2$ which will play an important role in the following.

The two-dimensional supersymmetry algebra (2.2) inherits a $U(1)$ R-symmetry from the $\mathcal{N}=1$ superalgebra in four dimensions under which $Q_\pm$ has charge +1 and $\bar{Q}_\pm$ has charge −1. We will call this symmetry $U(1)_R$. If the central charge vanishes, then the two-dimensional superalgebra has an additional R-symmetry, denoted $U(1)_A$, which corresponds to spatial rotations in the two reduced dimensions. The $U(1)_R \times U(1)_A$ charges of the supersymmetry generators can be represented as

$$\begin{array}{cc} \bar{Q}_+ & Q_- \\ & \\ \bar{Q}_- & Q_+\,, \end{array} \tag{2.3}$$

where generators in the (left-) right-handed column have $U(1)_R$ charge (−1) +1. Generators in the (bottom) top row have $U(1)_A$ charge (−1) +1.

In addition to the continuous R-symmetries described above the $\mathcal{N}=(2,2)$ supersymmetry algebra also has a discrete mirror automorphism which interchanges the supercharges Q_+ and $\bar{Q}_+$ and does not act on Q_- and $\bar{Q}_-$. This symmetry also interchanges the R-symmetry groups $U(1)_R$ and $U(1)_A$ Clearly, this is only an automorphism of the algebra (2.2), if the central charge Z vanishes. If $Z \neq 0$ the same transformation maps (2.2) to a mirror algebra in which a different central charge appears as the anti-commutator of Q_- and Q_+. The new central charge breaks $U(1)_R$ but leaves $U(1)_A$ is unbroken.

Multiplets of $\mathcal{N}=1$ supersymmetry in four dimensions yield multiplets of $\mathcal{N}=(2,2)$ supersymmetry after dimensional reduction to two dimensions. However, as we discuss below, not all $\mathcal{N}=(2,2)$ multiplets can be obtained in this way. Following Witten [8], it is convenient to start from an $\mathcal{N}=1$ superspace in four dimensions with coordinates X_m, θ_α, $\bar\theta_{\dot\alpha}$. The four-dimensional chiral multiplet correponds an $\mathcal{N}=1$

[3] Like the 4D spinor index, the 2D Dirac index is raised and lowered with the antisymmetric tensor $\epsilon^{12} = -\epsilon_{12} = 1$. Thus $(\psi^-, \psi^+) = (\psi^1, \psi^2)$ with $\psi^- = \psi_+$, $\psi^+ = -\psi_-$ and similar relations for $\bar\psi^\pm$.

chiral superfield $\Phi(X,\theta,\bar{\theta})$ which obeys the constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$. In the standard notation of Wess and Bagger [22], this superfield has the component expansion

$$\Phi = \phi(Y) + \sqrt{2}\theta^\alpha\psi_\alpha(Y) + \theta^\alpha\theta_\alpha F(Y)\,, \tag{2.4}$$

where $Y^m = X^m + i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$. The component fields include a complex scalar ϕ, a left-handed Weyl fermion ψ_α and a complex auxiliary field F. Similarly, the anti-chiral superfield $\bar{\Phi}$ obeys the constraint $D_\alpha\bar{\Phi} = 0$ and its component fields are the charge conjugate degrees of freedom $\bar{\phi}$, $\bar{\psi}_{\dot{\alpha}}$ and $\bar{F}$.

As above, we dimensionally reduce by considering only superfield configurations which are independent of X_1 and X_2. In the notation introduced above, this yields a two-dimensional chiral superfield $\Phi(x,\theta,\bar{\theta})$ which obeys the constraints $\bar{D}_-\Phi = \bar{D}_+\Phi = 0$. The two-dimensional anti-chiral superfield obeys the constraints $D_-\bar{\Phi} = D_+\bar{\Phi} = 0$. Thus the four-dimensional (anti-)chiral multiplet reduces to a two-dimensional (anti-)chiral multiplet with the same scalar field content and a two-dimensional Dirac fermion with components $\psi_\pm$ $(\bar{\psi}_\pm)$. The $U(1)_R \times U(1)_A$ charges of the dynamical fields can be represented as

$$\begin{array}{ccccc} & \bar{\psi}_+ & & \psi_- & \\ \bar{\phi} & & & & \phi \\ & \bar{\psi}_- & & \psi_+ & \end{array} \tag{2.5}$$

where, as before, columns (rows) correspond to $U(1)_R$ $(U(1)_A)$ charge. In this case the $U(1)_R$ $(U(1)_A)$ charges ranges from -2 to $+2$ $(-1$ to $+1)$. In fact the assignment of $U(1)_R$ charges is more subtle than indicated above: it is possible to redefine the R-symmetry generators by adding to them the generators of other global symmetries [8]. However this will not play an important role in the following.

The two basic invariant Lagrangians for two-dimensional chiral superfields Φ_i are obtained by dimensional reduction of their counterparts in four dimensions. These are the D-term Lagrangian[4]

$$\mathcal{L}_D = \int d^4\theta\, K(\Phi_i,\bar{\Phi}_i)\,, \tag{2.6}$$

where the Kähler potential, K, is a real function of Φ_i and $\bar{\Phi}_i$ and the F-term Lagrangian,

$$\mathcal{L}_F = \int d^2\theta\, W(\Phi_i) \;+ \int d^2\bar{\theta}\, \bar{W}(\bar{\Phi}_i)\,, \tag{2.7}$$

where the superpotential W is holomorphic in Φ_i. Both the D-term and the F-term are invariant under $U(1)_A$. The F-term is only invariant under $U(1)_R$ if the superpotential has charge $+2$ under this symmetry (modulo the subtlety about possible redefinitions of the $U(1)_R$ generator mentioned above). The bosonic terms in $\mathcal{L}_B + \mathcal{L}_F$ are given by

$$\mathcal{L}_{\rm Bose} = g_{i\bar{j}}\left(-\partial_\mu\phi^i\partial^\mu\bar{\phi}^j + F^i\bar{F}^j\right) + \left(F^i\frac{\partial W}{\partial\phi^i} + \bar{F}^i\frac{\partial\bar{W}}{\partial\bar{\phi}^i}\right)\,, \tag{2.8}$$

[4]The conventions for superspace integration are those of [22]. In 2D notation these read $d^2\theta = d\theta_-d\theta_+/2$, $d^2\bar{\theta} = d\bar{\theta}_-d\bar{\theta}_+/2$ and $d^4\theta = d^2\theta d^2\bar{\theta}$.

JHEP11(1998)005

where the Kähler metric $g_{i\bar{j}}$ is defined as

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial \phi^i \partial \bar{\phi}^j} \,. \tag{2.9}$$

The $\mathcal{N} = 1$ gauge multiplet in four dimensions consists of the gauge field V_m, right- and left-handed Weyl spinors[5] χ_α and $\bar{\chi}_{\dot{\alpha}}$ and a real auxiliary field D. These fields are components of a real superfield $V(X, \theta, \bar{\theta})$. After imposing the Wess-Zumino gauge condition, V has the expansion

$$V = -\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} V_m - \frac{1}{\sqrt{2}} \left(\theta^\alpha \theta_\alpha \bar{\theta}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} - \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \theta^\alpha \chi_\alpha \right) + \frac{1}{2} \theta^\alpha \theta_\alpha \bar{\theta}_{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} D \,. \tag{2.10}$$

After dimensional reduction, the four dimensional gauge field V_m yields a two-dimensional gauge field v_μ with $v_0 = V_0$ and $v_1 = V_3$ and a complex scalar $\sigma = V_1 - iV_2$. The Weyl fermions in the four-dimensional gauge multiplet reduce to two Dirac fermions with components $\chi_\pm$ and $\bar{\chi}_\pm$ respectively. The $U(1)_R \times U(1)_A$ charges of these fields are

$$\begin{array}{ccc} & \sigma & \\ \bar{\chi}_+ & & \chi_- \\ & v_\mu & \\ \bar{\chi}_- & & \chi_+ \\ & \bar{\sigma} & \end{array} \,. \tag{2.11}$$

A chiral multiplet can be coupled to the gauge field with the minimal coupling prescription,

$$\Phi \to \exp(V) \Phi \,. \tag{2.12}$$

The resulting Lagrangian for the component fields is given in in Appendix A.

In two dimensions it also possible to construct $\mathcal{N} = (2, 2)$ multiplets, known as twisted chiral multiplets [23], which are chiral with repect to supercharges Q_- and $\bar{Q}_+$. The corresponding twisted chiral superfield $\Lambda(x, \theta, \bar{\theta})$ obeys the constraints $D_- \Lambda = \bar{D}_+ \Lambda = 0$. This possiblility, which has no analog in four-dimensions, is closely related to the mirror automorphism of the supersymmetry algebra (2.2) which interchanges the supercharges Q_+ and $\bar{Q}_+$. To exhibit the symmetry between chiral and twisted chiral superfields it is convenient to introduce a twisted chiral superspace notation in which the fermionic coordinates θ_+ and $\bar{\theta}_+$ are interchanged. Thus we define twisted fermionic coordinates ϑ_α and $\bar{\vartheta}_{\dot{\alpha}}$ with $(\vartheta_1, \vartheta_2) = (\theta_-, \bar{\theta}_+)$ and $(\bar{\vartheta}_{\dot{1}}, \bar{\vartheta}_{\dot{2}}) = (\bar{\theta}_-, \theta_+)$. The twisted chiral superfield $\Lambda(x, \theta, \bar{\theta})$ has the expansion

$$\Lambda = \lambda(\mathcal{Y}) + \sqrt{2} \vartheta^\alpha \tilde{\rho}_\alpha(\mathcal{Y}) + \vartheta^\alpha \vartheta_\alpha E(\mathcal{Y}) \,, \tag{2.13}$$

[5]The fermion field χ_α introduced above is related to the fermion field λ_α of Wess and Bagger by $\chi_\alpha = -\sqrt{2} i \lambda_\alpha$. The unconventional normalization is chosen so that the formalism is symmetric between chiral and twisted chiral supermultiplets which will be defined below.

JHEP11(1998)005

where $\mathcal{Y}^m = X^m + i\vartheta^\alpha \sigma^m_{\alpha\dot\alpha}\bar\vartheta^{\dot\alpha}$. The component fields include a complex scalar λ, a 2D fermion $\tilde\rho_\alpha$ with $(\tilde\rho_1, \tilde\rho_2) = (\rho_-, \bar\rho_+)$ and a complex auxiliary field E. Similarly the twisted anti-chiral superfield $\bar\Lambda$ obeys the constraint $\bar D_- \bar\Lambda = D_+ \bar\Lambda = 0$ and has component fields $\bar l$, $\bar{\tilde\rho}_{\dot\alpha}$ and $\bar L$, where $(\bar{\tilde\rho}_{\dot 1}, \bar{\tilde\rho}_{\dot 2}) = (\bar\rho_-, \rho_+)$. The $U(1)_R \times U(1)_A$ charges of the dynamical fields of these multiplets are given as

$$\begin{array}{ccc} & \lambda & \\ \bar\rho_+ & & \rho_- \\ \bar\rho_- & & \rho_+ \\ & \bar\lambda & \end{array} \tag{2.14}$$

An important example of a twisted chiral multiplet is the multiplet associated with the gauge field strength. The abelian field strength $f = \epsilon^{\mu\nu}\partial_\mu v_\nu$ is contained in a gauge-invariant superfield $\Sigma = \bar D_+ D_- V$ whose lowest component is the complex scalar σ. By construction Σ obeys $\bar D_+ \Sigma = D_- \Sigma = 0$ and is therfore twisted chiral. In addition to σ, the twisted chiral multiplet also contains the fermion components χ_- and $\bar\chi_+$ and the complex field $S = D - if$. With the chosen normalization for the component fields (see footnote 5 above), Σ has an expansion of the standard form (2.13),

$$\Sigma = \sigma(\mathcal{Y}) + \sqrt{2}\vartheta^\alpha \tilde\chi_\alpha(\mathcal{Y}) + \vartheta^\alpha\vartheta_\alpha S(\mathcal{Y}) \,, \tag{2.15}$$

where, in the twisted superspace notation introduced above, the Dirac fermion $\tilde\chi_\alpha$ has components $(\tilde\rho_1, \tilde\rho_2) = (\rho_-, \bar\rho_+)$. Similarly one can define a twisted anti-chiral multiplet which contains the complex scalars $\bar\sigma$ and $D + if$ and the fermions $\bar\chi_-$ and χ_+. The corresponding twisted anti-chiral superfield, $\bar\Sigma$, obeys the constraints $\bar D_- \bar\Sigma = D_+ \bar\Sigma = 0$.

Invariant Lagrangians for twisted chiral superfields Λ_A can be constructed in direct analogy with those for conventional chiral superfields. The D-term Lagrangian (2.6) for chiral superfields is defined as an integral over all four of the fermionic coordinates. As the superspace measure $d^4\theta$ is invariant (up to an overall sign) under the interchange of θ_+ and $\bar\theta_+$, an invariant D-term for twisted chiral superfields can be constructed in exactly the same way,

$$\tilde{\mathcal{L}}_D = \int d^4\theta \, \mathcal{K}(\Lambda_A, \bar\Lambda_A) \,, \tag{2.16}$$

where $\mathcal{K}$ is a real function of Λ_A and $\bar\Lambda_A$. On the other hand the F-term Lagrangian (2.7) needs to be modified for twisted chiral superfields by interchanging the integrations over θ_- and $\bar\theta_+$. Thus we have a twisted F-term,

$$\tilde{\mathcal{L}}_F = \int d^2\vartheta \mathcal{W}(\Lambda_A) \; + \int d^2\bar\vartheta \, \bar{\mathcal{W}}(\bar\Lambda_A) \,, \tag{2.17}$$

where the integration measures are $d^2\vartheta = d\theta_- d\bar\theta_+/2$ and $d^2\bar\vartheta = d\bar\theta_- d\theta_+/2$. Note also that $d^4\theta = -d^2\vartheta d^2\bar\vartheta$ The twisted superpotential $\mathcal{W}$ is holomorphic in the twisted chiral superfields Λ_A. Both the D-term and the twisted F-term are invariant under $U(1)_R$, the F-term violates $U(1)_A$ unless the twisted superpotential has charge +2 under this symmetry. An example of a twisted superpotential which obeys this condition arises for the field strength superfield Σ. The Fayet-Iliopoulos D-term and the topological θ-term for the gauge field multiplet can be combined in the form [8],

$$\begin{aligned} \tilde{\mathcal{L}}_F &= \frac{i}{2}\int d^2\vartheta\, \tau\Sigma \;-\; \frac{i}{2}\int d^2\bar{\vartheta}\, \bar{\tau}\bar{\Sigma} \\ &= -rD + \frac{\theta}{2\pi} f\,, \end{aligned} \tag{2.18}$$

where the FI coupling r and the vacuum angle θ combine to form a complex coupling $\tau = ir + \theta/2\pi$. The resulting F-term Lagrangian is a special case of the general expression (2.17) with twisted superpotential $\mathcal{W} = i\tau\Sigma/2$.

Several general features of the vacuum structure and spectrum of effective theories which contain only twisted chiral superfields will be important in the following. Consider a theory of M twisted chiral superfields Λ_A $A = 1, 2\ldots, M$ with a low-energy effective Lagrangian of the form $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_D + \tilde{\mathcal{L}}_F$. In particular, we will consider the case of a generic superpotential which breaks $U(1)_A$ symmetry. The bosonic terms in the Lagrangian are

$$\tilde{\mathcal{L}}_{\rm Bose} = g_{A\bar{B}}\left(-\partial_\mu\lambda^A\partial^\mu\bar{\lambda}^B + E^A\bar{E}^B\right) + \left(E^A\frac{\partial\mathcal{W}}{\partial\lambda^A} + \bar{E}^A\frac{\partial\bar{\mathcal{W}}}{\partial\bar{\lambda}^A}\right). \tag{2.19}$$

With the conventions chosen above, the Kähler metric for twisted superfields is given by

$$g_{A\bar{B}} = -\frac{\partial^2\mathcal{K}}{\partial\lambda^A\partial\bar{\lambda}^B}\,. \tag{2.20}$$

After eliminating the auxiliary fields E_A the bosonic Lagrangian becomes

$$\tilde{\mathcal{L}}_{\rm Bose} = -g_{A\bar{B}}\partial_\mu\lambda^A\partial^\mu\bar{\lambda}^B - g^{A\bar{B}}\frac{\partial\mathcal{W}}{\partial\lambda^A}\frac{\partial\bar{\mathcal{W}}}{\partial\bar{\lambda}^B}\,, \tag{2.21}$$

where $g^{A\bar{B}}$ is the inverse Kähler metric, $g_{A\bar{C}}g^{\bar{C}B} = \delta_A^{\ B}$.

The condition for a supersymmetric vacuum is the vanishing of the potential energy,

$$U = g^{A\bar{B}}\frac{\partial\mathcal{W}}{\partial\lambda^A}\frac{\partial\bar{\mathcal{W}}}{\partial\bar{\lambda}^B} = 0\,. \tag{2.22}$$

In a unitary theory coordinates may be chosen so that $g^{A\bar{B}}$ is positive definite and thus the vacuum condition becomes $\partial\mathcal{W}/\partial\lambda_A = 0$ for $A = 1, 2, \ldots, M$. This provides M complex equations for the vacuum values of λ_A which constitute M complex unknowns. For a generic twisted superpotential $\mathcal{W}$, these conditions will have a finite number of isolated solutions.

In two dimensions topologically stable solitons or kinks potentially occur in any theory with two or more isolated vacuum states. These are static solutions of the classical field equations which asymptote to two different vacua at left and right spatial infinity. Specifically we consider time-independent field configurations with boundary conditions $\lambda_A \to \alpha_A$ as $x \to -\infty$ and $\lambda_A \to \beta_A$ as $x \to +\infty$, where α_A and β_A are two different solutions of the vacuum condition (2.22). The mass of such a configuration obeys the following inequality [24] which hold for any complex constant γ with $|\gamma| = 1$,

$$
\begin{aligned}
M &= \int_{-\infty}^{+\infty} dx \left[g_{A\bar{B}} \frac{\partial \lambda^A}{\partial x} \frac{\partial \bar{\lambda}^B}{\partial x} + g^{A\bar{B}} \frac{\partial \mathcal{W}}{\partial \lambda^A} \frac{\partial \bar{\mathcal{W}}}{\partial \bar{\lambda}^B} \right] \\
&= \int_{-\infty}^{+\infty} dx \left| \frac{\partial \lambda^A}{\partial x} - \gamma g^{A\bar{B}} \frac{\partial \bar{\mathcal{W}}}{\partial \bar{\lambda}^B} \right|^2 + \int_{-\infty}^{+\infty} dx \left[\bar{\gamma} \frac{\partial \mathcal{W}}{\partial \lambda^A} \frac{\partial \lambda^A}{\partial x} + \gamma \frac{\partial \bar{\mathcal{W}}}{\partial \bar{\lambda}^A} \frac{\partial \bar{\lambda}^A}{\partial x} \right] \\
&\geq \quad 2\mathrm{Re} \left[\bar{\gamma} \left(\mathcal{W}(\beta^A) - \mathcal{W}(\alpha^A) \right) \right] .
\end{aligned} \tag{2.23}
$$

By choosing $\gamma = \Delta\mathcal{W}/|\Delta\mathcal{W}|$ with $\Delta\mathcal{W} = \mathcal{W}(\beta^A) - \mathcal{W}(\alpha^A)$ we obtain the Bogomol'nyi bound $M \geq 2|\Delta\mathcal{W}|$.

As usual the Bogomol'nyi bound corresponds to a non-zero value for the central charge Z appearing in the supersymmetry algebra (2.2). Thus we have $Z = 2\Delta\mathcal{W}$. This is consistent with our choice of a twisted F-term which breaks the $U(1)_A$ symmetry. In fact, this formula for the central charge is generic to all $\mathcal{N} = (2,2)$ supersymmetric theories [4], even those which do not have an effective Lagrangian of the simple Landau-Ginzburg form $\tilde{\mathcal{L}}_D + \tilde{\mathcal{L}}_F$. In all cases, it is possible to define an effective twisted superpotential $\mathcal{W}_{\rm eff}$ which has critical points at each vacuum and contributes to the central charge as $Z = \Delta\mathcal{W}_{\rm eff}$. However, as explained in [1], there may also be additional contributions to the central charge from the generators of unbroken abelian global symmetries.

From (2.23) we deduce that a BPS saturated soliton solution satisfies the first order equation,

$$
\frac{\partial \lambda^A}{\partial x} = \frac{\Delta\mathcal{W}}{|\Delta\mathcal{W}|} g^{A\bar{B}} \frac{\partial \bar{\mathcal{W}}}{\partial \bar{\lambda}^B} . \tag{2.24}
$$

A useful property of such BPS saturated solutions can be deduced by multiplying both sides of this equation by $\partial\mathcal{W}/\partial\lambda^A$,

$$
\frac{\partial \mathcal{W}}{\partial x} = \frac{\partial \mathcal{W}}{\partial \lambda^A} \frac{\partial \lambda^A}{\partial x} = \frac{\Delta\mathcal{W}}{|\Delta\mathcal{W}|} g^{A\bar{B}} \frac{\partial \mathcal{W}}{\partial \lambda^A} \frac{\partial \bar{\mathcal{W}}}{\partial \bar{\lambda}^B} = \frac{\Delta\mathcal{W}}{|\Delta\mathcal{W}|} U(\mathcal{W}) . \tag{2.25}
$$

As the potential energy U is real, this implies that a BPS soliton configuration corresponds to a straight line segment in the complex $\mathcal{W}$ plane joining the points $\mathcal{W}(\alpha^A)$ and $\mathcal{W}(\beta^A)$. From the Bogomol'nyi bound, the mass of this soliton is $2|\Delta\mathcal{W}|$ which is the proportional to the length of the line segment.

3. The classical theory

As in section 1, we will consider a superrenormalizable $U(1)$ gauge theory with gauge superfield V and field strength Σ which is a twisted chiral superfield. The theory considered will also contain N chiral superfields Φ_i, $i = 1, 2, \ldots, N$ each of charge $+1$. A theory with only this chiral matter content would have a non-cancelling gauge anomaly in four-dimensions. However, in two dimensions the requirements for gauge anomaly cancellation are much less strict and the given matter content yields a consistent theory. The kinetic terms and minimal couplings for each of these fields can be written as a D-term in $\mathcal{N} = 2$ superspace,

$$\mathcal{L}_D = \int d^4\theta \left[\sum_{i=1}^{N} \bar{\Phi}_i \exp(2V)\Phi_i - \frac{1}{4e^2}\bar{\Sigma}\Sigma \right]. \tag{3.1}$$

The $U(1)$ gauge coupling e has the dimensions of a mass, which means that the kinetic terms for the gauge multiplet are irrelevant in the infra-red [8]. As reviewed in the previous section, the Fayet-Iliopoulis term and and topological θ-term can be combined in the twisted F-term,

$$\mathcal{L}_F = \int d^2\vartheta\, \mathcal{W}(\Sigma) \; + \int d^2\bar{\vartheta}\, \bar{\mathcal{W}}(\bar{\Sigma}) \,, \tag{3.2}$$

with the twisted superpotential $\mathcal{W} = i\tau\Sigma/2$. In two dimensions, the complexified coupling $\tau = ir + \theta/2\pi$ is dimensionless and corresponds to a marginal operator. In fact we will see that τ behaves very much like the complexified coupling $\tau_{4D} = i4\pi/g^2_{4D} + \theta_{4D}/2\pi$ in four-dimensional gauge theory. Both the F-term and the D-term are invariant under the full R-symmetry group, $U(1)_A \times U(1)_R$, at the classical level. The theory also has a global $SU(N)$ symmetry which acts on the flavour index i. In the following we will introduce $\mathcal{N} = (2, 2)$ supersymmetric mass terms for the chiral multiplets, Φ_i. However, we first consider the vacuum structure of the massless theory.

Eliminating the auxiliary fields by their equations of motion, the classical potential energy is

$$U = \sum_{i=1}^{N} |\sigma|^2 |\phi_i|^2 + e^2 \left(\sum_{i=1}^{N} |\phi_i|^2 - r \right)^2 . \tag{3.3}$$

For a supersymmetric vacuum both terms in U must vanish. For $r > 0$, the vanishing of the second term requires that at least one ϕ_i must be non-zero. The vanishing of the first term then requires that $\sigma = 0$. On the other hand, if $r = 0$, then we must have $\phi_i = 0$ for $i = 1, 2, \ldots N$ and then σ is not constrained. In the former case the $U(1)$ gauge invariance is spontaneously broken while in the latter case it is preserved. For this reason we will refer to these two sets of vacua as the classical Higgs branch and classical Coulomb branch respectively although the usual idea of a moduli space of inequivalent vacua does not apply in two dimensions. If $r < 0$, then there is no solution to the condition $U = 0$ and, at least at the classical level, there are no supersymmetric vacua.

The classical Higgs branch consists of the solution space of the equation

$$\sum_{i=1}^{N} |\phi_i|^2 = r \,, \tag{3.4}$$

modulo $U(1)$ gauge transformation which rotate each ϕ_i by the same phase; $\phi_i \to \exp(i\alpha)\phi_i$. This is precisely the definition of the complex projective space CP^{N-1}. The gauge degrees of freedom and the modes of the chiral fields which are orthogonal to the vacuum manifold acquire mass $\sqrt{r}e$ by the Higgs mechanism. In the low-energy limit $e \to \infty$, the kinetic term for the gauge multiplet vanishes, and the correponding component fields can be eliminated by their equations of motion. The resulting effective theory for the modes tangent to the vacuum manifold is an $\mathcal{N} = (2,2)$ supersymmetric σ-model with target space CP^{N-1}. The CP^{N-1} target is covered by N overlapping coordinate patches, $\mathcal{P}_j$ with $j = 0, 1, \ldots, N$, each with $N-1$ gauge-invariant complex coordinates $w_i^{(j)} = \phi_i/\phi_j$ with $i \neq j$. The patch $\mathcal{P}_j$ covers the complement in CP^{N-1} of the submanifold defined by $\phi_j = 0$. In this patch, an unconstrained form of the σ-model action can be obtained by using the constraint (3.4) to eliminate ϕ_j. The superspace Lagrangian can be written in terms of chiral superfields, $W_i^{(j)}$, whose scalar components are $w_i^{(j)}$,

$$\mathcal{L}_{\mathrm{eff}} = r \int d^4\theta \, \log\left(1 + \sum_{i=1}^{N}{}' \, \bar{W}_i^{(j)} W_i^{(j)}\right), \tag{3.5}$$

where $\sum'$ indicates that the term with $i = j$ in the sum is omitted. The superspace integrand is precisely the Kähler potential for the Fubini-Study metric on CP^{N-1}. Note that, in choosing particular a coordinate patch on the target space, we have concealed the $SU(N)$ symmetry of the model. The FI coupling of the underlying gauge theory is related to the dimensionless coupling constant of the σ-model as $r = 2/g^2$. Thus the low-energy σ-model is weakly coupled for $r >> 1$. The FI parameter r can also be thought of as the radius of the target space.

We will now consider how this analysis is modified in the presence of explicit mass terms for the chiral fields. The conventional supersymmetric mass term for chiral multiplets, which is familiar from four-dimensions, has the form of a superpotential bilinear in chiral superfields. In the present case, each of the chiral multiplets has the same charge and a mass term of this kind would violate gauge invariance. However, as pointed out recently by Hanany and Hori [1], there is another way of introducing a mass term which has no analog in four dimensions. In two dimensions the $\mathcal{N} = (2,2)$ gauge multiplet contains a complex scalar σ. On the classical Coulomb branch, for $\sigma \neq 0$, the potential (3.3) indicates that the chiral fields each acquire a mass $|\sigma|$ by the Higgs mechanism. Thus one may introduce a supersymmetric mass term for the chiral fields by coupling them to a background gauge multiplet in which the scalar field is frozen to its vacuum expectation value. As in [1], these will be refered to as twisted mass term. The new parameters can be introduced in a manifestly supersymmetric way by including a new gauge multiplet $\hat{V}_i$ for each twisted mass and constraining the corresponding field strengths $\hat{\Sigma}_i$ to the constant values m_i by integrating over Lagrange multipliers which are themselves twisted chiral superfields.

The resulting D-term Lagrangian for N chiral superfields of equal $U(1)_G$ charge with twisted masses m_i is,

$$\mathcal{L}_D = \int d^4\theta \left[\sum_{i=1}^{N} \bar{\Phi}_i \exp(2V + 2\langle\hat{V}_i\rangle)\Phi_i - \frac{1}{4e^2}\bar{\Sigma}\Sigma \right]. \tag{3.6}$$

In the four-dimensional superspace notation of section 1, the constant gauge background superfield, $\langle\hat{V}_i\rangle$, is given by

$$\langle\hat{V}_i\rangle = -\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \hat{V}_{mi}\,, \tag{3.7}$$

where $\hat{V}_{1i} = \mathrm{Re}(m_i)$, $\hat{V}_{2i} = -\mathrm{Im}(m_i)$ and $\hat{V}_{0i} = \hat{V}_{3i} = 0$. Note that only the differences between twisted masses are physically significant, the sum of the twisted masses can be absorbed by a constant shift in the complex scalar σ. Thus, without loss of generality, we may set $\sum_{i=1}^{N} m_i = 0$. The introduction of twisted mass terms affects the classical symmetries of the model. As the twisted masses are the scalar components of a background gauge multiplet, they carry $U(1)_A$ charge $+2$. Hence, in the presence of twisted masses, $U(1)_A$ is explicitly broken to Z_2 while $U(1)_R$ remains unbroken. Generic twisted masses with $m_i \neq m_j$ also break the global $SU(N)$ flavour symmetry of the massless theory down to its maximal abelian subgroup $U(1)^{N-1} = (\otimes_{j=1}^{N} U(1)_j)/U(1)_G$. As in section 1, the Φ_i have charges δ_{ij} under $U(1)_j$

In the presence of twisted masses, the potential energy becomes,

$$U = \sum_{i=1}^{N} |\sigma + m_i|^2 |\phi_i|^2 + e^2 \left(\sum_{i=1}^{N} |\phi_i|^2 - r \right)^2 . \tag{3.8}$$

The vacuum structure for the Higgs branch with $r > 0$ is now changed as follows. As before at least one chiral field, say ϕ_k, must be non-zero for the second term to vanish. Now, for the first term to vanish also, we must have $\sigma = -m_k$. However, if we have generic twisted masses with $m_i \neq m_j$, this also requires $\phi_i = 0$, for all $i \neq k$. Exactly one chiral field is non-zero in a given supersymmetric vacuum. It follows that there are exactly N supersymmetric vacua $\mathcal{V}_k$ labelled by $k = 1, 2, \ldots, N$. Thus one effect of the introduction of twisted masses is to lift the classical vacuum moduli space, CP^{N-1}, leaving N isolated vacua. As will be discussed below, this is consistent with the Witten index [25] for the CP^{N-1} σ-model which is equal to N. In the vacuum $\mathcal{V}_k$ the scalar fields take values $\sigma = -m_k$ and $|\phi_i| = \sqrt{r}\delta_{ik}$. Hence there is exactly one vacuum $\mathcal{V}_k$ in each coordinate patch $\mathcal{P}_k$.

As in the massless case, the low-energy effective action is obtained by taking the limit $e \to \infty$ and eliminating the gauge multiplet fields by their equations of motion. In the theory with twisted masses, this leads to a classically massive variant of the CP^{N-1} σ-model. To describe the theory in the vacuum $\mathcal{V}_j$, we choose the coordinate patch $\mathcal{P}_j$ with coordinates $w_i^{(j)}$ for $i \neq j$. The effective superspace Lagrangian becomes

$$\mathcal{L}_{\mathrm{eff}} = r \int d^4\theta \, \log \left(1 + \sum_{i=1}^{N}{}' \, \bar{W}_i^{(k)} \exp\left(2\langle\hat{V}_i\rangle - 2\langle\hat{V}_j\rangle\right) W_i^{(k)} \right), \tag{3.9}$$

where, as in (3.5), $W_i^{(j)}$ are chiral superfields with scalar components $w_i^{(j)}$. The background twisted superfields $\langle\hat{V}_i\rangle$ are defined in (3.7) above.

To illustrate the properties of the effective theory with twisted masses, it will be useful to consider the simplest non-trivial case, $N = 2$. In this case the low-energy theory involves a single chiral superfield, W, whose lowest component is the complex scalar field $w = \phi_1/\phi_2$. The component fields of W are minimally coupled to a background gauge field $\langle\hat{V}_1\rangle - \langle\hat{V}_2\rangle$ whose scalar component is $m = m_1 - m_2$. An explicit expression for the low-energy effective Lagrangian in terms of component fields is given in Appendix A. The bosonic part of the effective Lagrangian is,

$$\mathcal{L}^{(0)} = -\frac{1}{\rho^2}\left[r\left(\partial_\mu\bar{w}\partial^\mu w + |m|^2|w|^2\right) + \frac{\theta}{2\pi i}\varepsilon^{\mu\nu}\partial_\mu\bar{w}\partial_\nu w\right], \tag{3.10}$$

where $\rho = 1+|w|^2$. For $m = 0$, (3.10) reduces to the standard Lagrangian of the CP^1 σ-model. For $m \neq 0$, the classical theory has two supersymmetric vacua which are located at $w = 0$ and $w = \infty$ respectively. An equivalent Lagrangian which is non-singular in the neighbourhood of $w = \infty$ can be obtained by the coordinate transformation $w \to 1/w$.

4. The classical BPS spectrum

As described in section 2, BPS states obey the exact mass formula $M = |Z|$ where Z is the central charge appearing in the $\mathcal{N} = (2, 2)$ supersymmetry algebra (2.2). In $\mathcal{N} = (2, 2)$ theories the central charge recieves a contribution from the twisted F-terms in the Lagrangian. Specifically the contribution involves the difference of the twisted superpotential evaluated at left and right spatial infinity, denoted $\Delta\mathcal{W}$. As in section 2, this corresponds to a topological charge carried by Bogomoln'yi saturated solitons. However, in theories which have unbroken abelian global symmetries the corresponding generators can also contribute to the central charge. In the classical theory described in the previous section the global $SU(N)$ symmetry is broken by the twisted masses m_i to the abelian subgroup generated by the charges S_i. The corresponding formula for the central charge proposed in [1] is,

$$Z = 2\Delta\mathcal{W} + i\sum_{i=1}^{N} m_i S_i\,. \tag{4.1}$$

In the following, this formula will be checked explicitly for the *classical* theory whose low-energy description is the massive version (3.9) of the supersymmetric CP^{N-1} σ-model. In particular, we will find that the classical theory has three different kinds of states which obey the BPS mass formula $M = |Z|$: those that carry only the global charges S_i, those that carry only topological charges T_i, and states which carry both kinds of charge. In this section we consider these three types of states in turn. In the subsequent sections, the relevance of this classical spectrum to the corresponding quantum theory will be explained.

4.1. Elementary quanta

The fields appearing in the massive σ-model Lagrangian (3.9) transform under the global symmetries $U(1)_i$ and their quanta therefore carry the corresponding charges. In particular, consider the $N = 2$ case described by (3.10) above. Expanding around the vacuum at $w = 0$, the tree-level spectrum includes a spinless particle of mass $M = |m| = |m_1 - m_2|$ corresponding to the complex scalar field $w = \phi_1/\phi_2$. This scalar also has fermionic superpartner of the same mass. Both these states also carry $U(1)$ charges $S_1 = -S_2 = +1$, and therefore satisfy the BPS mass relation

$$M = |m_1 - m_2| = |S_1 m_1 + S_2 m_2| = |Z|\,. \tag{4.2}$$

Similarly, in the other supersymmetric vacuum $w = \infty$, the tree-level spectrum includes a BPS saturated multiplet whose scalar component has the quantum numbers of $w^{-1} = \phi_2/\phi_1$.

The above analysis can easily be generalized for arbitrary N. In this case the theory has N supersymmetric vacua $\mathcal{V}_i$ $i = 1, 2, \ldots, N$. In the vacuum $\mathcal{V}_j$, the low-energy theory is described by the superspace Lagrangian (3.9) which contains $N - 1$ chiral multiplets $W_i^{(j)}$ with $i \neq j$, described by the superspace Lagrangian (3.9). The particles corresponding to these fields have charge $S_i = +1$ and $S_j = -1$ and $S_k = 0$ for $k \neq i, j$. Expanding the fields around their values in the vacuum $\mathcal{V}_j$ one finds that the component fields of $W_i^{(j)}$ have tree-level masses $M_{ij} = |m_i - m_j|$. Hence the corresponding states satisfy the BPS mass relation

$$M_{ij} = |m_i - m_j| = \left|\sum_{i=1}^{N} m_i S_i\right| = |Z|\,. \tag{4.3}$$

In each of the N supersymmetric vacuum states, the tree-level spectrum includes $N - 1$ BPS saturated multiplets (and their anti-particles). Including the BPS states from each vacuum, there is one BPS multiplet corresponding to each of the $N(N-1)$ gauge-invariant fields $W_i^{(j)} = \Phi_i/\Phi_j$. In the next section we will consider quantum corrections to the classical BPS spectrum. One of the main results will be that the low-energy effective theory is weakly coupled for large values of the twisted masses. Hence, at least in this regime, the $N(N-1)$ multiplets found above are present in the BPS spectrum of the quantum theory.

4.2. Solitons

In general a two dimensional theory with isolated vacua can have topologically stable solitons or kinks. These are solutions of the classical equations of motion which interpolate between two different vacua at left and right spatial infinity. In the present case, where there are N isolated vacua $\mathcal{V}_i$, it is useful to define N topological charges T_i as follows. A soliton which asymptotes to $\mathcal{V}_k$ at $x = -\infty$ and $\mathcal{V}_l$ at $x = +\infty$ has charges

$T_k = -1$, $T_l = +1$ and $T_i = 0$ for $i \neq k, l$. According to (4.1), the BPS mass formula for this soliton is

$$M_{kl} = 2|\Delta\mathcal{W}| = 2\left|\sum_{i=1}^{N} \mathcal{W}(\sigma = -m_i)T_i\right|, \tag{4.4}$$

where the twisted superpotential is given by its classical value, $\mathcal{W} = i\tau\sigma/2$. Thus

$$\begin{aligned} M_{kl} &= 2|\mathcal{W}(\sigma = -m_l) - \mathcal{W}(\sigma = -m_k)| \\ &= |m_k - m_l|\sqrt{r^2 + \left(\frac{\theta}{2\pi}\right)^2}. \end{aligned} \tag{4.5}$$

This suggests that, for generic twisted masses, the classical theory includes BPS solitons with $N(N-1)/2$ different masses.

It is straightforward to find explicit forms for these soliton solutions. In the $N = 2$ case, this may be accomplished by performing a change of variables in the bosonic Lagrangian (3.10). Specifically it is convenient to decompose the complex field w in terms of its modulus and argument as

$$w = \tan\frac{\varphi}{2}\exp(i\alpha), \tag{4.6}$$

where, in order to make the mapping one-to-one, we make the identifications $\varphi \sim \varphi + 2\pi$ and $\alpha \sim \alpha + 2\pi$. In terms of the new variables, the bosonic Lagrangian reads

$$\mathcal{L}_{\text{Bose}} = -\frac{r}{4}\left[(\partial_\mu\varphi)^2 + \sin^2\varphi\left(|m|^2 - (\partial_\mu\alpha)^2\right)\right] + \frac{\theta}{4\pi}\epsilon^{\mu\nu}\partial_\mu(\cos\varphi)\partial_\nu\alpha\,. \tag{4.7}$$

In this form, the bosonic sector of the model is a close relative of the sine-Gordon (SG) scalar field theory. In the present case there is an additional massless field α with derivative couplings to the SG field φ. Very similar, but not identical, generalizations of the SG theory have been considered in the literature before [26]. The two SUSY vacua found above correpond to the two sets of zeros of the SG potential, which occur at $\varphi = 2n\pi$ and at $\varphi = (2n+1)\pi$ for integer n. As α appears only through its derivatives it can take any constant value in the vacuum.

The soliton solutions we seek can now be found explicitly by a trivial modification of the standard analysis of the sine-Gordon theory. The mass functional for time-dependent field configurations obeys the following inequalities:

$$\begin{aligned} M &= \frac{r}{4}\int_{-\infty}^{+\infty} dx \left(\frac{\partial\varphi}{\partial x}\right)^2 + \sin^2\varphi\left[\left(\frac{\partial\alpha}{\partial x}\right)^2 + |m|^2\right] \\ &\geq \frac{r}{4}\int_{-\infty}^{+\infty} dx \left(\frac{\partial\varphi}{\partial x} \pm |m|\sin\varphi\right)^2 \mp 2|m|\sin\varphi\frac{\partial\varphi}{\partial x} \\ &\geq \left|\frac{r}{4}\int_{-\infty}^{+\infty} dx 2|m|\sin\varphi\frac{\partial\varphi}{\partial x}\right| \;=\; \left|\frac{rm}{2}\left[\cos\varphi\right]_{-\infty}^{+\infty}\right|. \end{aligned} \tag{4.8}$$

Hence defining the topological charge, T,

$$T = -\frac{1}{2}\left[\cos\varphi\right]_{-\infty}^{+\infty}, \tag{4.9}$$

we have the Bogomol'nyi bound $M \geq r|m||T|$. Configurations which saturate the bound must have α =constant and φ a solution of the sine-Gordon equation,

$$\frac{\partial\varphi}{\partial x} = \pm|m|\sin\varphi\,. \tag{4.10}$$

In particular, the soliton which interpolates between two neighbouring vacua has $T = T_1 = -T_2 = 1$ and is given by

$$\varphi = \varphi_S(x;|m|) = 2\tan^{-1}\left[\mathrm{e}^{|m|x}\right], \tag{4.11}$$

where, for later convenience, we have introduced a notation which emphasizes the parametric dependence of the kink solution ϕ_S on $|m|$. The corresponding solution for w has the simple form $w = w_S = \exp(|m|x)$. In addition to the usual degeneracy associated with spatial translations, the $T = 1$ solution has an extra one-parameter degeneracy corresponding to the constant value of α. This reflects the fact that the soliton solution breaks the $U(1)$ global symmetry generated by $S = (S_1 - S_2)/2$. The mass of the solution is given by the Bogomol'nyi formula; $M = r|m|T = r|m|$. As $m = m_1 - m_2$, this agrees with the BPS mass formula (4.5) in the case $\theta = 0$. However, the analysis given above is not altered in the presence of a non-zero vacuum angle and thus does not agree with the BPS mass formula (4.5) for $\theta \neq 0$. In fact the correct θ dependence of the soliton mass will be recovered only after a more precise semiclassical analysis given below which includes the possibility of time-dependent classical solutions.

Finding the $N(N-1)$ BPS saturated soliton solutions for the general case $N > 2$ is now easy. The $N = 2$ static solution found above can trivially be embedded in the appropriate CP^1 subspace of CP^{N-1}. Explicitly, for a soliton which interpolates between the vacua $\mathcal{V}_k$ and $\mathcal{V}_l$ at left and right spatial infinity, we set $\phi_i = 0$ for $i \neq k, l$ and the bosonic Lagrangian reduces to the $N = 2$ expression (3.10) with $w = \phi_k/\phi_l$ and $m = |m_k - m_l|$. This yields a soliton solution of mass $M_{kl} = r|m_k - m_l|$. Again this agrees with the BPS mass formula in the case $\theta = 0$.

4.3. Dyons

In the previous section we discovered that the soliton solution of the low-energy effective theory which interpolates between the k'th and the l'th vacuum has a degeneracy associated with global $U(1)$ rotations generated by $S = (S_l - S_k)/2$. The soliton therefore has a periodic collective coordinate, $\alpha \in [0, 2\pi]$ which is analogous to the $U(1)$ 'charge-angle' of a BPS monopole in four-dimensions. In the four-dimensional case [27], allowing the collective coordinate to become time-dependent yields dyonic solutions which carry both magnetic and electric charges. This suggests that we should look for solutions of the equations of motion with time-dependent collective coordinate, $\alpha(t)$, which carry global $U(1)$ charge in addition to their topological charge. In the case $N=2$ the corresponding conserved charge is determined by Noether's theorem to be

$$S = \frac{r}{2}\int_{-\infty}^{+\infty} dx\, \sin^2\varphi\, \dot{\alpha} + \frac{\theta}{4\pi}\int_{-\infty}^{+\infty} dx\, \partial_x(\cos\varphi)\,. \tag{4.12}$$

The first term confirms that solutions with non-zero $\dot{\alpha}$ will generically have non-zero charge S. The signifigance of the second term will be explained below.

For the $N=2$ case, finding the desired time-dependent solutions is almost trivial: setting $\alpha=\omega t$ in the classical equation of motion simply has the effect of shifting the mass parameter $|m|^2 \to |m|^2-\omega^2$. Hence, in the notation introduced in (4.11) above, the solution for φ is,

$$\varphi = \varphi_\omega = \varphi_S(x; \sqrt{|m|^2-\omega^2}) \tag{4.13}$$

or, equivalently $w=\exp(\sqrt{|m|^2-\omega^2}\, x + i\omega t)$. Setting $\theta=0$ for the moment, the mass and charge of this solution may be evaluated to give,

$$M = \frac{r|m|^2}{\sqrt{|m|^2-\omega^2}}\,, \qquad S = \frac{r\omega}{\sqrt{|m|^2-\omega^2}}\,. \tag{4.14}$$

Evidently the solution only makes sense for $\omega<|m|$. Physically this correspond to the threshold above which the rotating soliton is unstable to the emission of soft quanta of the charged field.

Classically the angular velocity ω and therefore the charge S can take any value. In the quantum theory, the elementary quanta of the field w carry charge $+1$ and thus we expect that S is quantized in integer units. In the weak-coupling limit of the quantum theory we perform a semiclassical quantization of the dyon system. In this case, the allowed values of the charge are determined by the Bohr-Sommerfeld quantization condition. For a time-dependent solution of period $\tau=2\pi/\omega$, the action-per-period, $\mathcal{S}_\tau$, is quantized according to the condition

$$\mathcal{S}_\tau + M\tau = 2\pi n\,, \tag{4.15}$$

where n is an integer The action-per-period of the dyon solution $\phi=\phi_\omega$, $\alpha=\omega t$ can be evaluated to obtain

$$\mathcal{S}_\tau + M\tau = \frac{2\pi r\omega}{\sqrt{|m|^2-\omega^2}} = 2\pi S\,. \tag{4.16}$$

Hence the Bohr-Sommerfeld quantization condition (4.15) implies that the allowed values of S are integers as expected. The semiclassical spectrum of the model includes an infinite tower of dyon states with topological charge $T=1$ labelled by an integer $U(1)$ charge S with masses, $M=|m|\sqrt{S^2+r^2}$. These two-dimensional dyons are close relatives of the Q-kinks studied in [15].

It is straightforward to repeat the argument for non-zero vacuum angle θ. Both the $U(1)$ charge of the periodic solution and its the action-per-period are shifted by the contribution of the final term in the Lagrangian (4.7)

$$S \to S - \frac{\theta}{2\pi}\,, \qquad \mathcal{S}_\tau \to \mathcal{S}_\tau - \frac{\theta}{2\pi}\,, \tag{4.17}$$

while the mass and period of the solution are unchanged. For non-zero θ, even the static soliton solution with $\omega = 0$ acquires a non-zero $U(1)_S$ charge. The net effect of the two shifts appearing in (4.17) above cancels in (4.15) and the Bohr-Sommerfeld condition again implies that the charge S is quantized in integer units. The mass spectrum of the dyons becomes,

$$M = |m|\sqrt{\left(S + \frac{\theta}{2\pi}\right)^2 + r^2}\,. \tag{4.18}$$

This result is an exact analog of the Witten effect [7] for dyons in four-dimensional gauge theories. In particular the state with $S = 0$ has precisely the θ-dependent mass predicted in (4.5) above. As θ increases from 0 to 2π the charge of each state is shifted by one-unit; $S \to S + 1$. As states with each integer value of S are present in the theory, the spectrum is invariant under this shift.

The classical spectrum of the $N = 2$ model consists of particle states labelled by two seperate charges: the global $U(1)$ charge S and the topological charge T. We can now deduce a simple mass formula which applies to all the states we have discussed so far: the elementary particles and anti-particles with (S,T) charges $(\pm 1, 0)$, the soliton and dyon states with charges $(S, 1)$ and the charge conjugate states with charges $(-S, -1)$.

$$M_{S,T} = |m|\sqrt{\left(S + \frac{\theta}{2\pi}T\right)^2 + r^2T^2}\,. \tag{4.19}$$

This is consistent with the BPS mass formula $M = |Z|$ with the central charge

$$Z = -mi\,(S + \tau T)\,, \tag{4.20}$$

where τ is the complexified coupling constant $ir + \theta/2\pi$. This analysis is easily extended to the case of general N. The soliton which interpolates between the vacua $\mathcal{V}_k$ and $\mathcal{V}_l$ carries topological charge $T = (T_l - T_k)/2 = +1$ and has a collective coordinate associated with global $U(1)$ rotations generated by $S = (S_l - S_k)/2$. After applying the Bohr-Sommerfeld quantization condition (4.15) we find an infinite tower of dyons with integral $U(1)$ charges. The dyon masses are given by (4.19) with the replacement $m \to m_l - m_k$

To summarize the results of this section, we have found that the classical BPS spectrum includes both elementary particles which carry the global $U(1)$ Noether charges S_i and solitons which interpolate between different vacua and carry topological charges T_i. The two sets of charges can be written as two N-component vectors $\vec{S} = (S_1, S_2, \ldots S_N)$

and $\vec{T} = (T_1, T_2, \ldots T_N)$. The spectrum includes all states with one non-zero charge vector (which can be either $\vec{S}$ or $\vec{T}$) of the form $\pm(0, \ldots, +1, \ldots, -1, \ldots, 0)$. For each (non-zero) allowed value of the topological charge $\vec{T}$, the spectrum also includes an infinite tower of dyons with global charge vector $\vec{S} = S\vec{T}$ where S can be any integer. The masses of all these states obey the BPS mass formula $M = |Z|$ with

$$Z = -i\vec{m} \cdot (\vec{S} + \tau\vec{T})\,, \tag{4.21}$$

where $\vec{m} = (m_1, m_2, \ldots, m_N)$.

The spectrum described above bears a striking resemblance to the classical spectrum of BPS states of an $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in four dimensions with gauge group $SU(N)$ (see, for example, the introduction of [28]). The four-dimensional theory has a dimensionless gauge coupling g_{4D} and vacuum angle θ_{4D} which can be combined in a single complex coupling $\tau_{4D} = 4\pi/g_{4D}^2 + i\theta_{4D}/2\pi$. The theory contains a complex scalar field A in the adjoint representation of the gauge group The vacuum expectation value of this field is an $N \times N$ matrix with complex eigenvalues a_i which obey $\sum_{i=1}^{N} a_i = 0$. For generic non-zero a_i, $SU(N)$ is broken to its maximal abelian subgroup $U(1)^{N-1}$ and the the gauge bosons corresponding to the $N(N-1)$ broken generators get masses by the Higgs mechanism. These states and their superpartners are charged under the $U(1)$ factors of the unbroken gauge group. The charges can be represented as vectors $\vec{q} = (q_1, q_2, \ldots, q_N)$ with integer entries q_i obeying $\sum_{i=1}^{N} q_i = 0$. There are two massive gauge bosons for each of the $N(N-1)$ possible charge vectors of the form $\vec{q} = \pm(0, \ldots, +1, \ldots, -1, \ldots, 0)$ The theory also contains BPS monopoles which carry the corresponding magnetic charges $\vec{h} = (h_1, h_2, \ldots, h_N)$ for integers h_i with $\sum_{i=1}^{N} h_i = 0$. For each $U(1)$ the unit of magnetic charge is determined by the Dirac quantization condition to be $4\pi/g_{4D}^2$. In these units, the allowed magnetic charge vectors also have the form $\vec{h} = \pm(0, \ldots, +1, \ldots, -1, \ldots, 0)$. Finally, for each allowed charge vector $\vec{h}$, the theory has an infinite tower of dyons which carry magnetic charge $\vec{h}$ and electric charge $\vec{q} = Q\vec{h}$, for each integer Q. Each of these states gives rise to a BPS multiplet of $\mathcal{N} = 2$ supersymmetry with mass $M = \sqrt{2}|\mathcal{Z}|$ where,

$$\mathcal{Z} = \vec{a} \cdot (\vec{q} + \tau_{4D}\vec{h}) \tag{4.22}$$

and $\vec{a} = (a_1, a_2, \ldots, a_N)$.

The correspondence between the two theories is such that the global $SU(N)$ symmetry of the CP^{N-1} σ-model corresponds to the $SU(N)$ gauge symmetry of the four-dimensional theory. The Noether charges $\vec{S}$ corresponds to the electric charge vector $\vec{q}$ while the topological charges $\vec{T}$ is the analog of the magnetic charge $\vec{h}$. Up to an overall normalization, the twisted masses $\vec{m}$ correspond to the vacuum expectation value $\vec{a}$ and the dimensionless coupling τ is identified with the four-dimensional coupling τ_{4D}. Each BPS multiplet in the two-dimensional theory has a counterpart in the four-dimensional theory. Note, however, that while the $\mathcal{N} = 2$ theory in four dimensions has $N-1$ massless photons, the two-dimensional theory has no massless particles. Also note that, so

JHEP11(1998)005

far, the correspondence between the two theories is strictly classical. In section 6, we will find a modified version of the correspondence which applies at the quantum level.

5. Quantum effects

In this section we will consider how quantum corrections affect the analysis of the low-energy theory given above. We start with the case of vanishing twisted masses, $m_i = 0$ where the low-energy effective theory for $r > 0$ is the supersymmetric CP^{N-1} σ-model (3.5). The σ model coupling g is related to the FI parameter as $g = \sqrt{2/r}$. In the weak-coupling regime $r >> 1$, one may apply perturbation theory in the σ-model coupling. At one loop, one finds a logarithmic divergence which can be removed by renormalizing the bare coupling constant g_0 as [29, 30]

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} - \frac{N}{8\pi}\log\left(\frac{M_{\rm UV}^2}{\mu^2}\right), \tag{5.1}$$

where $g(\mu)$ is the renormalized coupling constant. Here M_{UV} is an ultra-violet regulator (for example a Pauli-Villars mass as in [30]) and μ is the RG subtraction point. It is shown in [29] that this is the only renormalization required to all orders in perturbation theory. The form of the one-loop correction (5.1) tells us that the model is asymptotically free. The converse to this fact is that the σ-model flows to strong coupling in the IR and hence perturbation theory is of no use in calculating low energy quantities such as the masses of particles. In particular perturbation theory breaks down at energy scales μ of the order of the RG invariant scale,

$$\Lambda = \mu\exp\left(-\frac{4\pi}{Ng^2(\mu)}\right). \tag{5.2}$$

As mentioned above, the true spectrum of the model bears no relation to the field content of the CP^{N-1} σ-model Lagrangian (3.5). While the classical theory contains massless degrees of freedom corresponding to gauge-invariant fields of the form $w_i = \phi_i/\phi_j$, the quantum theory has a mass-gap. For example, in the case $N = 2$ the true spectrum consists of two massive particles which form a doublet of the $SU(2)$ flavour symmetry and there are no asymptotic states with the quantum numbers of the field $w = \phi_1/\phi_2$ [13].

Another quantum effect which appears at one-loop in the massless theory is an anomaly in the $U(1)_A$ symmetry. The divergence of the $U(1)_A$ current has a non-vanishing contribution from a one-loop 'diangle' diagram. The resulting violation of $U(1)_A$ charge is,

$$\Delta Q_A = \frac{N}{\pi}\int d^2x\, f_{\rm E} = 2Nk\,, \tag{5.3}$$

where $f_{\rm E}$ is the Euclidean field strength of the gauge field and k is the topological charge. Thus $U(1)_A$ is explicitly broken to Z_{2N} by the anomaly. As usual the anomaly means that the vacuum angle θ can be set to zero by a $U(1)_A$ rotation of the fields and is no longer a physical parameter of the theory. Both the 2D gauge theory and the σ-model to which it reduces at low energy, have instantons with integer topological charge. The anomaly formula (5.3) reflects the fact that the k-instanton solution of the CP^{N-1} σ-model has a total of $2Nk$ fermion zero modes. Each of these zero modes carry one unit of $U(1)_A$ charge. Two zero modes correspond to the action of the supersymmetry generators which act non-trivially on the instanton solution and two more correspond to the action of superconformal generators. The remaining zero modes do not correspond to symmetries.

In the light of the classical correspondence discussed in the previous section, it is interesting to compare the two quantum effects described above with the quantum behaviour of the corresponding $\mathcal{N}=2$ theory in four dimensions. In the four-dimensional theory, the gauge coupling runs logarithmically at one loop. However, the coefficient of the logarithm which appears in the four-dimensional version of (5.1) is exactly twice that of the two-dimensional theory. In addition, the four-dimensional theory also has an anomalous $U(1)$ R-symmetry. A charge-k $SU(N)$ Yang-Mills instanton in the minimal $\mathcal{N}=2$ theory has $4Nk$ fermion zero modes which is twice the number of the corresponding σ-model instanton. Hence, the violation of R-charge in the background of a four-dimensional instanton is twice that given in (5.3). Equivalently the four-dimensional anomaly breaks the $U(1)$ R-symmetry down to Z_{4Nk} compared to the residual Z_{2Nk} of the two-dimensional theory. In fact, both these discrepancies can be resolved by considering a four-dimensional theory with N additional hypermultiplets rather than the minimal $\mathcal{N}=2$ theory. In this case the renormalization of the σ-model coupling is precisely the same as that of the gauge coupling. The residual discrete R-symmetries of the two theories also agree. This correspondence will be refined below.

In the case of non-zero twisted masses $m_i \neq 0$, the analysis of the quantum theory is somewhat different. As described above, the low-energy theory is now a variant of the CP^{N-1} σ-model with an explicit mass term for each of the scalar fields w_i (and superpartners) which appear in the classical action (3.9). As in the massless case, a divergent one-loop diagram leads the logarithmic renormalization of the σ-model coupling given in (5.1). However, now the particles running around the loop have masses $|m_i - m_j|$. As before we will start with the simplest case $N=2$ where there is a single massive charged particle with mass $|m| = |m_1 - m_2|$. For energy scales much less than $|m|$ the massive particle decouples and the σ-model coupling is frozen at a fixed value $g_{\text{eff}} \simeq g(\mu = |m|)$. This result, which will be confirmed in an explicit one-loop calculation in the soliton background below, means that the low-energy theory is weakly coupled for $|m| >> \Lambda$. For $N > 2$, the various particles appearing in the tree-level spectrum can have different masses and the story is more complicated; the heaviest particle decouples at the highest energy scale leaving an effective theory with one less flavour and so on. However the fact remains that if all the bare masses $|m_i - m_j|$ are much larger than the RG-invariant scale Λ, then $g_{\text{eff}} << 1$ and the low-energy theory is weakly coupled. Thus we come to the opposite conclusion to the one reached above for the massless case: perturbation theory applied to the massive σ-model action (3.9) should be reliable.

According to the above discussion of the case $N=2$, low-energy quantities such as the classical soliton mass, $M_0 = r|m|$, which depend on the FI parameter at the classical level should be corrected at one loop by the replacement,

$$r \to r_{\rm eff}(|m|) = \frac{1}{2\pi}\log\left(\frac{|m|^2}{\Lambda^2}\right). \tag{5.4}$$

As this effect is an important ingredient in understanding the BPS spectrum of the model an explicit derivation of this result is given below. In any theory, the one-loop correction to the mass of a soliton comes from performing a Gaussian path-integral over the fluctuations of the fields around their classical values. Explicitly the one-loop correction is given by [31]

$$M_1 = M_0 + \frac{\hbar}{2}\left(\sum\lambda_{\rm B} - \sum\lambda_{\rm F}\right) + \Delta M_{\rm ct}\,, \tag{5.5}$$

where $\lambda_{\rm B}$ and $\lambda_{\rm F}$ are the frequencies of the normal modes for small fluctuations of the bose and fermi fields respectively. $\Delta M_{\rm ct}$ is the contribution of the one-loop counter-term, which appears in the renormalization (5.1) of the σ-model coupling, evaluated on the soliton background. Explicitly, for $N=2$, the Lagrangian counter-term corresponding to (5.1) is

$$\Delta\mathcal{L}_{\rm ct} = \frac{\delta r}{r}\mathcal{L} = \frac{1}{\pi r}\log\left(\frac{M_{\rm UV}}{\mu}\right)\mathcal{L}\,. \tag{5.6}$$

Thus the contiribution of the counterterm to the one-loop soliton mass is

$$\Delta M_{\rm ct} = \int_{-\infty}^{+\infty} dx\Delta\mathcal{L}_{\rm ct}[\varphi_S(x)] = \frac{\delta r}{r}M_{\rm cl}\,. \tag{5.7}$$

As the counter-term contribution is divergent, the only way to get a finite answer for the soliton mass is if the sum appearing in (5.5) is also divergent and the two divergences cancel.

In order to determine the frequencies $\lambda_{\rm B}$ and $\lambda_{\rm F}$, we need to expand the Lagrangian to quadratic order in each of the fields around the classical solution $\varphi_{\rm cl} = \varphi_S(x)$, $\alpha_{\rm cl} = 0$. The fluctuating fields can be written as two real scalar fields $\vec{u}^{\rm T} = (u_1, u_2)$ and two Majorana fermions $\vec{\rho}_\alpha^{\,\rm T} = (\rho_{1\alpha}, \rho_{2\alpha})$, with $\vec{\rho}_\alpha^{\,\dagger} = \vec{\rho}_\alpha^{\,T}$. As in section 2, the index α runs over the two values $-$ and $+$.

$$\mathcal{L} = \mathcal{L}[\varphi_{\rm cl}, \alpha_{\rm cl}] - \frac{r}{4}\left(\vec{u}\cdot\mathcal{M}_{\rm B}\cdot\vec{u} + \vec{\rho}_\alpha^{\,T}\cdot\mathcal{M}_{\rm F}^{\alpha\beta}\cdot\vec{\rho}_\beta\right). \tag{5.8}$$

With a judicious choice of basis for the fluctuating fields which is given explicitly in Appendix B, the differential operators $\mathcal{M}_{\rm B}$ and $\mathcal{M}_{\rm F}$, simplify to

$$\mathcal{M}_{\rm B}^{ij} = \delta^{ij}\left(\frac{\partial^2}{\partial t^2} + \Delta_{\rm B}\right), \qquad \mathcal{M}_{\rm F}^{ij} = \delta^{ij}\begin{pmatrix} \frac{\partial}{\partial t} & D \\ -D^T & \frac{\partial}{\partial t}\end{pmatrix}, \tag{5.9}$$

with

$$D = \frac{\partial}{\partial x} + m \cos\left(\varphi_S(x)\right) \qquad D^T = -\frac{\partial}{\partial x} + m \cos\left(\varphi_S(x)\right) \tag{5.10}$$

and

$$\Delta_{\mathrm{B}} = DD^T = -\frac{\partial^2}{\partial x^2} + |m|^2 \cos\left(2\varphi_S(x)\right). \tag{5.11}$$

In this basis, the fluctuating degrees of freedom consists of two-decoupled copies of the fluctuations around the soliton of the supersymmetric sine-Gordon model and the analysis from this point on is an easy generalization of the calculation given by Kaul and Rajaraman [32]. Separating out the time dependence of the bosonic fluctuations as $\vec{u} = \vec{\mu} \exp(i\lambda_{\mathrm{B}} t)$ the resulting eigenvalue problem we need to solve is

$$\Delta_{\mathrm{B}} \vec{\mu}(x) = \lambda_{\mathrm{B}}^2 \vec{\mu}(x). \tag{5.12}$$

The operator Δ_{B} appears in the corresponding calculation in the sine-Gordon model and its spectrum is well known. In particular, it has a single normalizable zero mode $u^{(0)}(x) = \varphi_S'(x)/\sqrt{M}$. Hence the vector $\vec{\mu}$ has two linearly-independent zero modes; $(u^{(0)}, 0)$ and $(0, u^{(0)})$. As usual, the bosonic zero-modes should correspond to collective coordinates of the soliton. As in any two-dimensional theory the configuration $\varphi_{\mathrm{cl}}, \alpha_{\mathrm{cl}}$ yields a one-parameter family of solutions under spatial translations $x \to x + X$. However, as explained in the previous section, there is also an additional degeneracy which is not present in the ordinary sine-Gordon model: the soliton has a periodic collective coordinate, α, which corresponds to global $U(1)$ rotations. Thus the number of collective coordinates matches the number of bosonic zero modes. In addition to the zero modes Δ_{B} has a continuous spectrum of scattering modes scattering eigenstates $u^{(k)}(x)$ with eigenvalues $\lambda_{\mathrm{B}} = \sqrt{k^2 + |m|^2}$ and phase-shift,

$$\delta_{\mathrm{B}}(k) = -2 \tan^{-1}\left(\frac{k}{|m|}\right). \tag{5.13}$$

The fluctuations, $\vec{\rho}_\alpha$, of the the fermi fields are governed by the equation $\mathcal{M}_{\mathcal{F}}{}^{\alpha\beta} \cdot \vec{\rho}_\beta = 0$. From (5.9), we see that this equation has time-dependent solutions of the form

$$\vec{\rho}_\pm = \vec{\alpha}_\pm \exp\left(-i\lambda_{\mathrm{F}} t\right) + \vec{\alpha}_\pm^* \exp\left(+i\lambda_{\mathrm{F}} t\right), \tag{5.14}$$

where $\vec{\alpha}_\pm$ satisfy the equations

$$iD\vec{\alpha}_+ = -\lambda_{\mathrm{F}} \vec{\alpha}_- \qquad -iD^T \vec{\alpha}_- = -\lambda_{\mathrm{F}} \vec{\alpha}_+ . \tag{5.15}$$

Acting on the first and second equation with $-iD^T$ and iD respectively we find

$$D^T D \vec{\alpha}_+ = \Delta_{\mathrm{B}} \vec{\alpha}_+ = \lambda_{\mathrm{F}}^2 \vec{\alpha}_+ \quad DD^T \vec{\alpha}_- = \left(-\frac{\partial^2}{\partial x^2} + |m|^2\right) \vec{\alpha}_- = \lambda_{\mathrm{F}}^2 \vec{\alpha}_- . \tag{5.16}$$

The spinor component $\vec{\alpha}_+$ obeys the same equation as the bosonic fluctuation $\vec{\mu}$. This component therefore has two normalizable zero modes and a continuum of scattering states labelled by the momentum k with $\lambda_{\rm F} = \sqrt{k^2+|m|^2}$. As above the scattering for momentum k occurs with a phase shift $\delta_{\rm F}^{+}$ given by (5.13). On the other hand, the other spinor component $\vec{\alpha}_-$ is an eigenstate of the free massive Klein-Gordon operator. This operator has no normalizable zero modes. It does have scattering eigenstates with $\lambda_{\rm F} = \sqrt{k^2+|m|^2}$ although, because there is no interaction, the corresponding phase shift is zero, $\delta_{\rm F}^{+}=0$.

Taken together, the above facts imply that the fermion fields have a net total of two normalizable zero modes; $\vec{\alpha}_+ = (\varphi'_S/\sqrt{M}, 0)$ and $\vec{\alpha}_+ = (0, \varphi'_S/\sqrt{M})$ with $\vec{\alpha}_- = 0$ in both cases. Like the bosonic zero modes, these modes can be interpreted in terms of the symmetries of the theory which are broke by the soliton solution. In particular, the fermion zero modes correspond to supersymmetry generators which act non-trivially on the soliton field configuration. The fact that there are only two zero modes is significant: it means that there are only two such generators. Conversely, the soliton solution must be invariant under two of the four supersymmetries of the theory. This confirms that the solitons of this model are BPS saturated.

Turning to the non-zero modes, the above analysis indicates that the bose and fermi fields have the same non-zero eigenvalues: a continuum starting at $\lambda_{\rm F} = |m|$. Naively, this suggests that the contributions of bose and fermi fluctuations should cancel in the one-loop correction to the soliton mass (5.5). In fact, as explained in [32], this reasoning is not correct. For a continuous spectrum of eigenvalues the sums appearing in (5.5) are replaced by integrals over a density of eigenvalues. Placing the system in a spatial box of length L, the density of states is given by the formula [32]

$$\rho(k) = \frac{d{\rm n}}{dk} = \frac{1}{2\pi}\left[L + \frac{d\delta}{dk}\right]. \tag{5.17}$$

As we saw above, one component of the fermion fluctuations has zero phase shift while the remaining component and the bose fluctuations both have the non-zero phase shift (5.13). Thus, although the non-zero eigenvalues of the Bose and Fermi fields are equal, the densities of these eigenvalues are not. The sums appearing in (5.5) can be evaluated as

$$\begin{aligned}\sum\lambda_{\rm B} - \sum\lambda_{\rm F} &= \int_{-\infty}^{+\infty} dk\,\sqrt{k^2+|m|^2}\,(\rho_{\rm B}(k)-\rho_{\rm F}(k)) \\ &= 2\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\sqrt{k^2+|m|^2}\left[\frac{d\delta_{\rm B}}{dk} - \frac{1}{2}\left(\frac{d\delta_{\rm F}^{+}}{dk}+\frac{d\delta_{\rm F}^{-}}{dk}\right)\right] \\ &= -4|m|\int_0^{\infty}\frac{dk}{2\pi}\,\frac{1}{\sqrt{k^2+|m|^2}}\,. \end{aligned}\tag{5.18}$$

The final integral is logarithmically divergent. Introducing a UV cut-off M_{UV} as the upper limit of this integral, the one-loop formula for the soliton mass becomes,

$$M_1 = M_{\rm cl} - \frac{m}{\pi} \log\left(\frac{M_{UV}}{|m|}\right) + \Delta M_{\rm ct}\,. \tag{5.19}$$

From (5.7) we have

$$M_1 = \left(1 + \frac{\delta r}{r}\right) M_{\rm cl} - \frac{|m|}{\pi} \log\left(\frac{M_{UV}}{|m|}\right) = |m| \left(r(\mu) - \frac{1}{\pi} \log\left(\frac{\mu}{|m|}\right)\right). \tag{5.20}$$

The μ dependence of the soliton mass can be eliminated using the definition (5.2) of the RG invariant scale Λ. The final result is

$$M_1 = \frac{|m|}{\pi} \left[\log\left(\frac{|m|}{\Lambda}\right) + C\right]. \tag{5.21}$$

As expected the classical and one-loop masses are related by the replacement (5.4) of the tree-level FI coupling by a low-energy effective FI coupling. In fact only the divergent parts of the one-loop correction were evaluated carefully in the above analysis. The undetermined numerical constant C reflects the possible contribution of finite counter-terms at one loop. In particular, to evaluate C, one would have to carefully compare the renormalization prescriptions used in the vacuum and one-soliton sectors. For a recent treatment of this issue see[6] [33].

The analysis of the $U(1)_A$ anomaly is also modified by the introduction of twisted masses. In the massless case, the $U(1)_A$ anomaly implied the vacuum angle, θ, could be shifted by a $U(1)_A$ rotation of the fields. In this case theories with different values of the vacuum angle θ are physically equivalent. Non-zero twisted masses, explicitly break $U(1)_A$ down to Z_2. However, as twisted masses correspond to the expectation values of background twisted chiral superfields, the symmetry can be restored by assigning $U(1)_A$ charge $+2$ to the masses. Hence a simultaneous rotation of the fields and the twisted masses is required to shift the vacuum angle. Specializing to the $N = 2$ case considered above, the anomaly means that theories with different values of θ are equivalent to theories with the same value of θ which differ in the complex phase $\exp(i\beta)$ of the mass parameter m. Taking into account the coefficient of the anomaly (5.3), this means that physical quantities should depend on θ and β in the combination $\theta_{\rm eff} = \theta - 2\beta$. In the previous section we discovered a two-dimensional analog of the Witten effect for BPS dyons: the global $U(1)$ charge, S, of the dyon was shifted by an amount $\theta/2\pi$. According to the above discussion we should replace θ by $\theta_{\rm eff}$ in the mass formula (4.18). Thus the spectrum of BPS states undergoes a non-trivial monodromy as we follow a

[6]Note however, that the model considered here has twice as many supersymmetries as the minimal supersymmetric theories considered in [12, 32]. In particular, the above theory has short multiplets and therefore quantum corrections must respect the BPS mass formula. In the case of the minimal theories this conclusion can be avoided as pointed out by [33].

JHEP11(1998)005

large circle in the complex m plane. In particular, as $m \to \exp(2\pi i)m$ the spectrum of dyons transforms as

$$(S,1) \to (S-2,1) \qquad (S,-1) \to (S+2,-1)\,. \tag{5.22}$$

As the elementary particle invariant under this transformation, the monodromy of a BPS state with charges (S,T) can be compactly written as $(S,T) \to (S-2T,T)$.

In fact it is straightforward to exhibit the above monodromy directly in the one-loop analysis of the quantum corrections to the soliton and dyon masses. As in the previous section we expand to quadratic order in fluctuations of the fields around the classical background. However, in this case the classical background is the time-dependent dyon solution, $\varphi_{\rm cl} = \varphi_\omega(x)$ and $\alpha_{\rm cl} = \omega t$. The resulting expansion of the Lagrangian is

$$\mathcal{L} = \mathcal{L}[\varphi_{\rm cl}, \alpha_{\rm cl}] - \frac{r}{4}\left(\vec{u}\cdot\mathcal{M}_{\rm B}\cdot\vec{u} + i\bar{\Psi}_\alpha\hat{\mathcal{M}}_{\rm F}^{\alpha\beta}\Psi_\beta\right). \tag{5.23}$$

In this case we have chosen to parametrize the fermionic fluctuations in terms of a single Dirac fermion Ψ rather than two Majorana fermions. The corresponding γ-matrices are $\gamma^0 = i\sigma_2$, $\gamma^1 = -\sigma_1$ and $\gamma^5 = \gamma^0\gamma^1 = -\sigma_3$. Details of the calculation leading to (5.23) are given in Appendix B. In this basis the differential operator $\hat{\mathcal{M}}_{\rm F}$ can be written as

$$\begin{aligned}\hat{\mathcal{M}}_{\rm F} &= \gamma^\mu\partial_\mu + \cos\left(\varphi_\omega(x)\right)\left[i\omega\gamma^0 + {\rm Re}(m)I + i{\rm Im}(m)\gamma^5\right] \\ &= \gamma^\mu(\partial_\mu + a_\mu) + \cos\left(\varphi_\omega(x)\right)|m|\exp\left(i\beta\gamma^5\right),\end{aligned} \tag{5.24}$$

where, as above, $\beta = \arg(m)$ and we have defined an auxiliary, two-dimensional gauge field a_μ which has components $a_0 = \omega\cos\left(\varphi_\omega(x)\right)$ and $a_1 = 0$. Further details of the expansion in fluctuations around the soliton background are given in Appendix B. At one loop, the dependence of dyon masses on the phase β can only come from the β dependence in the formula (5.24) for $\hat{\mathcal{M}}_{\rm F}$. The one-loop contribution of fermion fluctuations to the effective action in the soliton sector is,

$$S_{\rm F} = \frac{1}{2}\log\det\left[\hat{\mathcal{M}}_{\rm F}\right], \tag{5.25}$$

where

$$\det{}^{\frac{1}{2}}\left[\hat{\mathcal{M}}_{\rm F}\right] = \int\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\exp\left[i\int d^2x\,\bar{\Psi}\hat{\mathcal{M}}_{\rm F}\Psi\right]. \tag{5.26}$$

The dependence of the exponent on the phase β can be absorbed by a chiral transformation of the Dirac fields;

$$\Psi \to \Psi' = \exp\left(i\frac{\beta}{2}\gamma^5\right)\Psi\,, \qquad \bar{\Psi} \to \bar{\Psi}' = \bar{\Psi}\exp\left(i\frac{\beta}{2}\gamma^5\right), \tag{5.27}$$

which gives

$$\bar{\Psi}\hat{\mathcal{M}}_{\rm F}\Psi_\beta = \bar{\Psi}'\hat{\mathcal{M}}_{\rm F}^{(0)}\Psi'_\beta\,, \tag{5.28}$$

with

$$\hat{\mathcal{M}}_{\rm F}^{(0)} = \gamma^\mu(\partial_\mu + a_\mu) + |m|\cos\left(\varphi_\omega(x)\right). \tag{5.29}$$

Naively it appears that by performing this change of variables in the path integral we can prove that the determinant appearing in (5.26) does not depend on β. However, this argument incorrectly assumes that the path integral measure is invariant under chiral rotations. The standard analysis of the chiral anomaly (see for example Chapter 9 of [34]) reveals that the measure transforms under (5.27) by a phase factor which depends on the topological charge of the background fields. For a two-dimensional Dirac fermion coupled to a background gauge-field a_μ, the measure transforms with a non-trivial Jacobian: $\mathcal{D}\Psi\mathcal{D}\bar{\Psi} = \mathcal{J}\mathcal{D}\Psi'\mathcal{D}\bar{\Psi}'$ where

$$\mathcal{J} = \exp\left[-i\beta\int d^2x\,\frac{1}{2\pi}\varepsilon^{\mu\nu}\partial_\mu a_\nu\right]. \tag{5.30}$$

Using the explicit form of the gauge field a_μ given above we have

$$= \exp\left[\frac{i\beta\omega}{2\pi}\int d^2x\,\partial_x\left(\cos(\varphi_S(x))\right)\right]. \tag{5.31}$$

The exponent of the Jacobian factor provides a one-loop correction to the action-per-period of the dyon solution,

$$\begin{aligned}\mathcal{S}_\tau &= \int_0^\tau dt\int_{-\infty}^{+\infty}dx\,\left[\mathcal{L}[\varphi_{\rm cl},\alpha_{\rm cl}] + \frac{\beta\omega}{2\pi}\partial_x\left(\cos(\varphi_S(x))\right)\right]\\ &= 2\pi S - M\tau - \theta + 2\beta T.\end{aligned} \tag{5.32}$$

As $T=1$ for the dyon, we find that θ enters in the Bohr-Sommerfeld quantization condition in the combination $\theta_{\rm eff}(m) = \theta - 2\beta$ as expected. This dependence leads to the non-trivial monodromy (5.22).

In the previous section we found that the $N=2$ theory, has a spectrum of BPS states with charges (S,T) which obey a mass formula $M = |Z|$ with $Z = -im(S+\tau T)$. In this section we have exhibited one-loop effects which replace the parameters r and θ which appear in the central charge by their effective counterparts $r_{\rm eff}$ defined in (5.4) and $\theta_{\rm eff} = \theta - 2\arg(m)$. These two effects combine to replace τ with

$$\tau_{\rm eff} = ir_{\rm eff} + \theta_{\rm eff}/2\pi = \frac{i}{\pi}\log\left(\frac{m}{\Lambda}\right). \tag{5.33}$$

Importantly $\tau_{\rm eff}$, and thus the central charge itself, is a holomorphic function of m and Λ. This holomorphic dependence could have been anticipated because both the twisted masses and the Λ-parameter correspond to the expectation values of background twisted chiral superfields. As discussed in section 1, the resuting contribution to the central charge corresponds to the difference between the asymptotic values of some twisted superpotential $\mathcal{W}_{\rm eff}$. which is holomorphic function of the twisted chiral superfields. In the next section the relevant twisted superpotential will be determined exactly.

So far we have determined the central charge to one-loop in the weak-coupling expansion of the low-energy σ-model (3.10). In the remainder of this section we will discuss the possible form of higher-order corrections. As discussed above, we expect that the exact central charge will have the form, $Z = -i(mS + m_D T)$ where $m_D = \Delta\mathcal{W}_{\text{eff}}$ is a holomorphic function of m and Λ which is approximately equal to $\tau_{\text{eff}} m$ for $|m| >> \Lambda$. The correspondence with $\mathcal{N} = 2$ theories in four-dimensions suggests that the exact form of m_D can be further constrained by the following arguments [35]. First, by RG invariance, the BPS masses can only depend on the FI coupling through the dynamical scale Λ. Holomorphy and dimensional analysis then imply that perturbative corrections in the σ-model coupling correspond to inverse powers of $\log(m/\Lambda)$. However, the weak-coupling monodromy indicates that m_D can only have the single branch-cut which arises at one-loop. The additional branch cuts introduced by higher-loop corrections would necessarily spoil the monodromy of the BPS spectrum discovered above. In particular, the spectrum of the theory would no longer be uniquely defined for given values of m and Λ. Thus, for self-consistency, all the higher-order perturbative corrections to the one-loop formula for m_D must vanish.

Next we consider possible non-perturbative corrections to the one-loop result for m_D given above. The $U(1)_A$ symmetry of the classical theory is restored if the twisted mass-parameter m is assigned charge $+2$. As m_D depends linearly on the effective twisted superpotential, it must also have charge $+2$ if the twisted F-term in the effective theory is to be $U(1)_A$ invariant. The classical result $m_D = \tau m$ is consistent with this charge assignement. In the quantum theory, $U(1)_A$ is explicitly broken to Z_4. The anomaly equation (5.3) implies that, in the sector of the theory with topological charge k, $U(1)_A$ is violated by $4k$ units. This corresponds to a correction to m_D of the form $m(\Lambda/m)^{2k}$. Using the explicit form of the Λ-parameter (5.2), this correction depends on the σ-model coupling as $\exp(-4\pi k/g^2)$. This is precisely the dependence expected for the leading semiclassical contribution of k σ-model instantons. In particular, because the introduction of a twisted mass breaks both $U(1)_A$ and the (super-)conformal invariance of the classical theory, it can lift fermion zero modes of the instanton. In general, the only modes which will not be lifted are the two modes which come from the action of supersymmetry generators on the instanton background. In this case the effective Fayet-Iliopoulis parameter of the theory can get corrections from arbitrary numbers of σ-model instantons. The presence of an explicit mass scale should also render the integral over the instanton scale-size convergent. Each of these features is analogous to the case of Yang-Mills instantons and their contribution to the effective gauge coupling in four-dimensional $\mathcal{N} = 2$ theories [35].

To summarize the main results of this section, the BPS mass formula the $N = 2$ theory is $M = |Z|$ where $Z = -i(mS + m_D T)$ where m_D is a holomorphic function of m and Λ with the weak coupling expansion,

$$m_D = \frac{im}{\pi}\left[\log\left(\frac{m}{\Lambda}\right) + \sum_{k=1}^{\infty} c_k \left(\frac{\Lambda}{m}\right)^{2k}\right], \tag{5.34}$$

where c_k are undetermined numerical coefficients. The first term in (5.34) comes from one-loop perturbation theory while the term proportional to c_k is a k-instanton contribution. In the next section we will find the exact formula for m_D and thereby determine the unknown coefficients c_k. Note that the form of (5.34) is consistent with the modified version of the classical correspondence described in the previous section: the quantum-corrected BPS spectrum implied by (5.34) has precisely the same form as that of $\mathcal{N} = 2$ supersymmetric gauge theory in four-dimensions with gauge group $SU(N)$ and N additional hypermultiplets in the fundamental representation. Specifically the parameters m and m_D correspond to the two BPS charge vectors a and a_D in the Seiberg-Witten solution of the four-dimensional theory. In the classical correspondence the complex coupling τ was identified its four-dimensional counterpart; $\tau_{4D} = i4\pi/g_{4D}^2 + \theta_{4D}/2\pi$. Similarly, at the quantum level, the Λ-parameters of the two theories are identified. This quantum correspondence will be made precise in the next section.

6. The exact BPS spectrum

So far we have analysed the two-dimensional gauge theory introduced in section 1 in a particular region of parameter space where it reduces to a massive variant of the supersymmetric CP^{N-1} σ-model. This requires that the dimensionful gauge coupling e is much larger than the dynamical scale Λ. The characteristic feature of this regime is that the $U(1)$ gauge symmetry is spontaneously broken and the gauge degrees of freedom get large masses from the Higgs mechanism. In this case the theory has instantons which give a complicated series of corrections to the BPS mass spectrum. For the case $N = 2$, the general form of these corrections, which involve contributions from all numbers of instantons, was given in (5.34) above. On the other hand, as discussed in section 1, the masses of BPS state are independent of the gauge-coupling and therefore we may attempt to calculate them in another more favourable region of parameter space. In the following, we will analyse the BPS spectrum in the case $e << \Lambda$ where the theory has very different behaviour from that described above. We will find that the $U(1)$ gauge symmetry is unbroken and the matter multiplets, rather than the gauge multiplet, are very massive. In this regime, the weakly-coupled description of the theory does not have instantons and an exact calculation of the BPS spectrum is possible.

As discussed above, the condition to decouple the gauge theory modes and obtain a CP^{N-1} σ-model at low energies is $e >> \Lambda$. In this context it is useful to think of the mass scale e as the effective UV cut-off for the low-energy σ-model. As in any asymptotically free theory, a large heirarchy between the cutoff and the Λ-parameter arises if the theory is weakly coupled at the cut-off scale. Thus the condition $e >> \Lambda$ can be written in terms of the renormalized σ model coupling (5.1) as $g^2(\mu = e) << 1$ or, in terms of the FI parameter, $r(\mu = e) >> 0$. In the following, we want to consider the theory in the opposite limit $e << \Lambda$, which according to (5.1) corresponds to the regime of *negative* FI parameter, $r(\mu = e) << 0$. We begin by considering the model without twisted masses. The classical bosonic potential given in section 3 was

$$U = \sum_{i=1}^{N} |\sigma|^2 |\phi_i|^2 + e^2 \left(\sum_{i=1}^{N} |\phi_i|^2 - r \right)^2 . \tag{6.1}$$

For $r < 0$, the states of minimum energy have $\phi_i = 0$ with arbitrary σ. However, the energy density of these minima does not vanish as is required for a supersymmetric vacuum state, but has the non-zero value $e^2 r^2$. Thus, at first sight, it appears that supersymmetry is spontaneously broken for $r < 0$. On the other hand, the Witten index of the $r > 0$ theory (or, equivalently, that of the CP^{N-1} σ-model) is equal to N. By standard arguments [25], this should preclude supersymmetry breaking for any value of r.

The resolution of this apparant paradox was given by Witten in [8]: in order to obtain the correct vacuum structure for $r < 0$, it is necessary to include quantum effects. In this case the minima of the potential (6.1) generically have $\sigma \neq 0$ as described above, and therefore the chiral multiplet scalars, ϕ_i, and their superpartners, gain masses equal to $|\sigma|$. In the region of field space $|\sigma| >> e$, these chiral multiplets become very massive and can be integrated out to give a low-energy effective action for the twisted superfield Σ with scalar component σ. In the same region of field space, the gauge coupling is small compared to the other mass scales in the theory and perturbation theory in the gauge coupling is reliable. Note that in the previous sections we considered a different weak-coupling expansion, namely perturbation theory in the σ-model coupling. Gauge theories in two dimensions are super-renormalizable and, in the present case, the only divergent diagram appears at one-loop [8]. This logarithmic divergence can be removed by a renormalization of the FI parameter r,

$$r(\mu) = r_0 - \frac{N}{4\pi} \log \left(\frac{M_{UV}^2}{\mu^2} \right) , \tag{6.2}$$

where M_{UV} is a UV cut-off and μ is the RG subtraction scale. Notice that the form of this renormalization is identical to that which occured in the $r > 0$ theory. In particular, just as in that case, we may eliminate the renormalized FI parameter $r(\mu)$ in favour of the RG invariant scale;

$$\Lambda = \mu \exp \left(-\frac{2\pi r(\mu)}{N} \right) . \tag{6.3}$$

Including terms with up to two derivatives or four fermions, the effective action has the form

$$\mathcal{L}_{\text{eff}} = \int d^4\vartheta \; \mathcal{K}_{\text{eff}}[\Sigma, \bar{\Sigma}] + \int d^2\vartheta \; \mathcal{W}_{\text{eff}}[\Sigma] + \int d^2\bar{\vartheta} \; \bar{\mathcal{W}}_{\text{eff}}[\bar{\Sigma}] . \tag{6.4}$$

After performing the renormalization of the FI parameter described above, the effective twisted superpotential is given at one loop by [36, 8],

$$\mathcal{W}_{\text{eff}} = \frac{i}{2} \Sigma \left(\hat{\tau} - \frac{N}{2\pi i} \log \left(\frac{2\Sigma}{\mu} \right) \right) . \tag{6.5}$$

In fact, as the massive fields which are integrated out to obtain (6.5) only appear quadratically in the action, the one-loop result is actually exact (see Appendix A of [4]). The complexified coupling constant $\hat{\tau}$ is equal to $ir(\mu)+\theta/2\pi+n^*$ where the integer n^* is chosen to minimize the potential energy. As explained in [8], this minimization of the potential reflects the fact that a non-zero value of the θ parameter in two-dimensions corresponds to a constant background electric field [37]. The states of the system with $n \neq n^*$, are unstable to pair creation of charged particles which screens the background field leaving the state with $N = n^*$. The corresponding potential energy is

$$U = g^{\Sigma\bar{\Sigma}} \left| \frac{\partial \mathcal{W}_{\text{eff}}}{\partial \sigma} \right|^2 , \tag{6.6}$$

where $g^{\Sigma\bar{\Sigma}} = (g_{\Sigma\bar{\Sigma}})^{-1}$ is the inverse of the Kähler metric

$$g_{\Sigma\bar{\Sigma}} = -\frac{\partial^2 \tilde{K}_{\text{eff}}}{\partial\Sigma\partial\bar{\Sigma}} . \tag{6.7}$$

Supersymmetric vacua correspond to the zeros of U. As the Kähler potential recieves only finite corrections of order e^2/σ^2, for large σ we can be confident that the inverse metric $g^{\Sigma\bar{\Sigma}}$ is non-vanishing. Thus the relevant condition for a supersymmetric vacuum is the vanishing of $\partial\mathcal{W}_{\text{eff}}/\partial\sigma$ which requires,

$$\sigma^N = \left(\frac{\mu}{2e}\right)^N \exp\left(2\pi i \tau(\mu)\right) = \tilde{\Lambda}^N , \tag{6.8}$$

where $\tilde{\Lambda} = \Lambda \exp(-1 + i\theta/N)/2$. Equation (6.8) has N solutions

$$\sigma = \tilde{\Lambda} \exp\left(\frac{2\pi i n}{N}\right) , \tag{6.9}$$

where $n = 1, 2, \ldots, N$. Thus the one-loop corrected effective theory has N supersymmetric vacua even though the classical theory had none. This number coincides with the known value of the Witten index of the supersymmetric CP^{N-1} σ-model. It also matches the number of classical vacua found above in the Higgs phase with non-zero twisted masses. As in the Higgs phase, the chiral anomaly breaks $U(1)_A$ down to Z_{2N}. Note however, that topological charge is not quantized in the Coulomb phase and the Euclidean field equations do not have instanton solutions [13]. Note also that the non-zero vacuum values for σ spontaneously break the residual Z_{2N} symmetry down to Z_2.

The theory has N isolated supersymmetric vacua and can therefore have BPS saturated solitons which interpolate between these vacua. As the low energy effective theory includes only twisted chiral superfields, we can immediately apply formulae given at the end of section 2. According to (2.23), a soliton solution interpolating between the vacua with $n = l$ and $n = k$ at left and right spatial infinity has mass $M = |Z|$ where,

$$Z = 2\Delta\mathcal{W}_{\text{eff}} = 2\left[\mathcal{W}_{\text{eff}}\left(\sigma = \exp\frac{2\pi i l}{N}\right) - \mathcal{W}_{\text{eff}}\left(\sigma = \exp\frac{2\pi i k}{N}\right)\right] . \tag{6.10}$$

Because of the spontaneously broken Z_{2N} symmetry, the resulting mass only depends on the difference $p = l - k$ and, for each p, is given by

$$M_k = \frac{N}{\pi}\left|\exp\left(\frac{2\pi i p}{N}\right) - 1\right|\tilde{\Lambda}\,, \tag{6.11}$$

one can also check that for $p = 1, 2 \ldots N$ the degeneracy of BPS states is,

$$D_p = \binom{N}{p}\,, \tag{6.12}$$

The lightest soliton states with $p = 1$ and degeneracy N are interpreted as the elementary quanta of the the chiral fields Φ_i [13]. In fact they carry charge $+1$ under the unbroken $U(1)$ gauge symmetry. These states transform in the fundamental representation of the flavour symmetry group $SU(N)$. The states with $p > 1$ correspond to stable bound states of p different flavours of elementary quanta and therefore transform in the p'th antisymmetric tensor representation of $SU(N)$. The degeneracy D_p in (6.12) is equal to the dimension of this representation and is therefore consistent with this interpretation. In contrast, the elementary quanta associated with the fields of the gauge multiplet are not BPS saturated [13].

The analysis given above depended on integrating out the chiral multiplets. This step is only valid if the masses of these fields are much larger than those of the gauge multiplet, which requires $|\sigma| >> e$. As the vacua in (6.9) are located at $\sigma \sim \Lambda$, the analysis is self-consistent for $e << \Lambda$, as advertised in section 1. In the opposite limit $e >> \Lambda$, the low-energy theory becomes the CP^{N-1} σ-model and the true vacuum states have $\phi_i \neq 0$ and spontaneously broken $U(1)$ gauge symmetry. Thus the theory moves from a Coulomb phase to a Higgs phase as the gauge coupling is increased. Despite this, there are several quantities in the theory which are independent of e. The simplest of these is the Witten index itself which is equal to N for all values of the parameters. The theory also has another supersymmetric index, introduced in [17], which is independent of D-term parameters such as e and depends holomorphically on twisted F-term parameters such as τ and m_i. Importantly, for the present purposes, these include the BPS mass spectrum. Thus the mass formula (6.11) is exact for all values of e and going to the regime $e >> \Lambda$, (6.11) gives the exact BPS spectrum of the supersymmetric CP^{N-1} σ-model. In particular, the mass spectrum agrees with that obtained by other methods which exploit the integrability of the model to obtain its exact spectrum and S-matrix [2, 38].

Only a small modification of this analysis is needed to include non-zero twisted masses [1]. The same reasoning as before leads to an effective twisted superpotential;

$$\mathcal{W}_{\text{eff}} = \frac{i}{2}\left(\hat{\tau}\Sigma - \frac{1}{2\pi i}\sum_{i=1}^{N}(\Sigma + m_i)\log\left(\frac{2}{\mu}(\Sigma + m_i)\right)\right), \tag{6.13}$$

Setting $\partial\mathcal{W}_{\rm eff}/\partial\Sigma=0$, the resulting condition for a supersymmetric vacuum state can be written as

$$\prod_{i=1}^{N}(\sigma+m_i)-\tilde{\Lambda}^N=\prod_{i=1}^{N}(\sigma-e_i)=0. \tag{6.14}$$

Thus there are N supersymmetric vacua at $\sigma=e_i$ for $i=1,\ldots,N$. Consider a soliton obeying the boundary conditions, $\sigma\to e_k$ as $x\to-\infty$ and $\sigma\to e_l$ as $x\to+\infty$. The soliton mass is given by $M_{kl}=|Z_{kl}|$ where a short calculation reveals that

$$\begin{aligned} Z_{kl} &= 2\left[\mathcal{W}_{\rm eff}(e_l)-\mathcal{W}_{\rm eff}(e_k)\right] \\ &= \frac{1}{2\pi}\left[N(e_l-e_k)-\sum_{i=1}^{N}m_i\log\left(\frac{e_l+m_i}{e_k+m_i}\right)\right]. \end{aligned} \tag{6.15}$$

At least for $|m_i-m_j|<<\Lambda$, the BPS property ensures that the soliton states of the CP^{N-1} σ-model persist in the massive case. Typically the only effect of introducing small non-zero values of $|m_i-m_j|$ is to introduce splittings between the degenerate states which form multiplets of $SU(N)$ in the massless limit. This reflects the fact that the twisted masses explicitly break the $SU(N)$ global symmetry down to its maximal abelian subgroup. More generally, the global description of the BPS spectrum may be complicated by the presence of curves of marginal stability in the parameter space on which the number of BPS states can change

Even for small twisted masses, the branch-cuts in the logarithms appearing in (6.15) lead to an ambiguity in the BPS spectrum. In particular the ambiguity in the central charge is equal to $i\sum_{N=1}^{\infty}m_in_i$ where the choice of integers n_i corresponds to a choice of branch for each of the N logarithms in (6.15). As explained by Hanany and Hori in [1], this ambiguity signals the fact that solitons can carry integer values of the global $U(1)$ charges S_i in addition to their topological charges. Clearly this is related to the existence of BPS dyons at weak-coupling discussed in the previous sections. As explained above, the BPS masses are independent of the gauge coupling e and therefore (6.15) should also provide an exact description of the mass spectrum even in the σ-model limit $e>>\Lambda$ In particular, the modified formula for the central charge

$$Z=2\Delta\mathcal{W}+\sum_{i=1}^{N}m_iS_i\,, \tag{6.16}$$

should apply for all values of the parameters. Indeed, this has exactly the same form as the formula (4.1) for the central charge of the classical theory considered in section 3; the two formulae differ only by the replacement of the classical twisted superpotential (2.17) by its quantum counterpart (6.13). In terms of the Noether and topological charges, $\vec{S}$ and $\vec{T}$, introduced in section 3, (4.1) becomes,

$$Z=-i(\vec{m}\cdot\vec{S}+\vec{m}_D\cdot\vec{T})\,, \tag{6.17}$$

with

$$\vec{m}_D=2i\left(\mathcal{W}_{\rm eff}(e_1),\mathcal{W}_{\rm eff}(e_2),\ldots\mathcal{W}_{\rm eff}(e_N)\right). \tag{6.18}$$

The exact mass formula for all BPS states in the theory is $M = |Z|$ with Z given by equations (6.17) and (6.18). In particular, for $|m_i - m_j| >> \Lambda$, this formula can be directly compared with the results of the semiclassical analysis given in the previous sections. As before, it is helpful to consider an explicit example for the case $N = 2$. Setting $m_2 = -m_1 = m/2$ the vacuum equation (6.8) becomes

$$\sigma^2 - \frac{m^2}{4} = \tilde{\Lambda}^2 \,. \tag{6.19}$$

Thus the two supersymmetric vacua are located at $\sigma = \pm\sqrt{m^2/4 + \tilde{\Lambda}^2}$. For $m = 0$, this reduces to $\sigma = \pm\tilde{\Lambda}$ which is a special case of (6.9). On the other had, for $|m| >> \Lambda$ the two vacua are approximately located at $\sigma = \pm m/2$ which agrees with the classical analysis of section 3. The central charge is $Z = -i(mS + m_D T)$ where $S = (S_1 - S_2)/2$, $T = (T_1 - T_2)/2$ and

$$m_D = -\frac{i}{\pi}\left[\sqrt{m^2 + 4\tilde{\Lambda}^2} + \frac{m}{2}\log\left(\frac{m - \sqrt{m^2 + 4\tilde{\Lambda}^2}}{m + \sqrt{m^2 + 4\tilde{\Lambda}^2}}\right)\right]. \tag{6.20}$$

In the weak-coupling limit, $m >> \tilde{\Lambda}$, this yields the expansion,

$$m_D = \frac{im}{\pi}\left[\log\left(\frac{m}{\tilde{\Lambda}}\right) + i\pi + \sum_{k=1}^{\infty} c_k \left(\frac{\tilde{\Lambda}}{m}\right)^{2k}\right], \tag{6.21}$$

with $c_k = (-1)^k(2k-2)!/(k!)^2$. The first term in (6.21) agrees with the one-loop formula for m_D derived in section 5 and, in principle, predicts the value of the undetermined additive constant C appearing in the one-loop soliton mass formula (5.21). The remaining series of terms have exactly the right form to be interpreted as the leading semiclassical contributions of σ-model instantons. The explicit result for the coefficients c_k constitutes an infinite set of predictions which could be tested against first-principles instanton calculations. The semiclassical analysis of the previous sections established that the weakly-coupled theory at large $|m|$ contains an infinite tower of dyon states with charge (S, T) where $T = \pm 1$ and S is an integer. As m describes a large circle in the complex plane the dyon spectrum under goes a non-trivial monodromy $(S, T) \to (S - 2T, T)$. The BPS spectrum also exhibits another phenomenon which is familiar in the context of $\mathcal{N} = 2$ theories in four dimensions: strongly coupled vacua with a massless dyon. Equation (6.20) shows that a dyon state becomes massless at $m = \pm 2i\tilde{\Lambda}$. As in $\mathcal{N} = 2$ SUSY Yang-Mills [5], each dyon can be made massless at one of these two points by successively applying the weak-coupling monodromy. We can choose conventions so that a dyon with even S can become massless at $m = +2i\tilde{\Lambda}$ while one with odd S becomes massless $m = -2i\tilde{\Lambda}$.

In addition to the dyon states, the weak coupling theory also has an elementary particle with $(S, T) = (1, 0)$. According to the above analysis, The mass of this state is equal to its tree level value $|m|$. Naively, this implies that the theory contains a massless particle for $m = 0$. However, for $m = 0$ we know that the theory reduces to the supersymmetric CP^1 σ-model which certainly has a mass gap. In fact, for $|m| << \Lambda$, the BPS spectrum should reduce to that of the σ-model which has only two stable BPS states as opposed to the infinite number of states present at weak coupling. As usual this discrepancy can be resolved if the two regions of parameter space are separated by a curve of marginal stability (CMS) on which the number of BPS states can change. In the following, at least for the case of the elementary particle, we will perform an explicit check that the required CMS exists. We will consider paths in the complex m-plane which connect the weak and strong coupling regions and show that there is at least one point on each path at which the elementary particle is exactly at threshold to decay. The paths in question are radial lines parametrized by $r > 0$ with $m/2i\tilde{\Lambda} = r\exp(i\delta)$ for a fixed value of $\delta \in [0, 2\pi]$. Rather than performing a complete check, we will restrict our attention to the simplest cases $\delta = 0,\pi$ and $\delta = \pi/2$, $3\pi/2$. At the end of this section, a more general argument for the existence of the required CMS will be given which exploits the connection between the BPS spectra considered here and those of $\mathcal{N} = 2$ theories in four dimensions.

For $\delta = 0$, the central charge determined by (6.20) becomes

$$Z = -i\left(m(S + \frac{1}{2}T) - \frac{2i}{\pi}\tilde{\Lambda}f\left(\frac{m}{2\tilde{\Lambda}}\right)T\right), \tag{6.22}$$

where

$$f(r) = \sqrt{1+r^2} + \frac{r}{2}\log\left(\frac{\sqrt{1+r^2}-r}{\sqrt{1+r^2}+r}\right) \tag{6.23}$$

is a continuous, real function of r for $r > 0$. The masses of states with (S, T) charges $(1,0)$, $(0,1)$ and $(1,-1)$ obey the inequality

$$M_{(1,0)} = m \leq 2\sqrt{\left(\frac{m}{2}\right)^2 + \frac{4\tilde{\Lambda}^2}{\pi^2}f^2\left(\frac{m}{2\tilde{\Lambda}}\right)} = M_{(0,1)} + M_{(1,-1)}\,, \tag{6.24}$$

with equality if and only if $f = 0$. Thus the elementary particle with charge $(1,0)$ is stable against decay into the dyon/soliton states with charges $(0,1)$ and $(1,-1)$ as long as $f \neq 0$. However, for $f = 0$ the elementary particle is exactly at the threshold for decay into these products. Hence, to prove that this path in parameter space passes through a CMS we must find a zero of $f(r)$ for some positive r. The asymptotic behaviour of $f(r)$ for large positive r is $f \sim -r\log r$ which is arbitrarily large and negative. On the other hand we have $f(0) = 1 > 0$. As f is continuous, the existence of the required zero follows immediately from the intermediate value theorem. A trivial modification of this argument shows that a similar point exists on the $\delta = \pi$ path. For $\delta = \pi/2$ we have $m/2\tilde{\Lambda} = i\nu$ for real positive ν. In this case, $m_D = 0$ and a dyon becomes massless at $\nu = 1$ and the elementary particle is at threshold to decay into the same products as the $\delta = 0$ case. The other singular point lies on the path with $\delta = 3\pi/2$ and a similar argument applies.

In section 4, we found a classical correspondence between the BPS states of the two-dimensional theory with N chiral multiplets and those of four-dimensional $\mathcal{N}=2$ SYM with gauge group $SU(N)$. In the following we will investigate whether this correspondence persists at the quantum level. The exact BPS spectrum of a quantum $\mathcal{N}=2$ theory in four dimensions is determined by the periods of an elliptic curve. We would therefore like to interpret the exact central charges (6.15) of the two-dimensional theory as periods of a complex curve and determine which, if any, four-dimensional theory is described by the same curve. Before presenting a curve with the required property, we briefly review our expectations of the corresponding four-dimensional theory. As explained at the end of the previous section, the naive extension of the classical correspondence found above is incorrect because the R-symmetry anomalies of the two theories in question are different. In order to match the pattern of R-symmetry breaking of the two-dimensional theory it is necessary to go to a four-dimensional $\mathcal{N}=2$ theory with gauge group $SU(N)$ and N additional hypermultiplets in the fundamental representation. To specify the correspondence we should determine the relation between the parameters (or moduli in the four-dimensional case) of the two theories. In the classical case, we identified the N complex eigenvalues, a_i, of the adjoint scalar VEV in the four-dimensional theory with the N twisted masses, m_i, of the two-dimensional theory. However, the story is more complicated for the proposed quantum correspondence as the four-dimensional theory with N hypermultiplets also has N additional parameters: the hypermultiplet masses μ_i. Clearly some additional conditions on the parameters/moduli of the four-dimensional theory are required to specify the correspondence.

In fact it is straightforward to find a complex curve whose non-vanishing periods are given by (6.15). For each N, the curve is defined by a polynomial equation of order $2N$ in x

$$y^2 = \frac{1}{4}\left[\prod_{i=1}^{N}(x+m_i) - \tilde{\Lambda}^N\right]^2 = \prod_{i=1}^{N}(x-e_i)^2\,. \qquad (6.25)$$

The roots, e_i are those defined in (6.8) above with the replacement $\sigma \to x$. As expected, this curve describes and $SU(N)$ $\mathcal{N}=2$ SQCD in four-dimensions with N hypermultiplets having masses $\mu_i = -m_i$. The four-dimensional theory is at a special point on its Coulomb branch where a maximal number of one-cycles vanish and N massless hypermultiplets [7] appear. This point is the root of the baryonic Higgs branch [18]. Correspondingly, the polynomial (6.25) is a perfect square; each of the N distinct roots e_i occurs exactly twice. By performing the change of variables

$$t = y - \frac{1}{2}\left[\prod_{i=1}^{N}(x+m_i) + \Lambda^N\right], \qquad (6.26)$$

[7]Note however that the BPS states of the two-dimensional theory correspond to the non-vanishing cycles of the curve and are all massive as expected.

the curve can be rewritten in an alternative form which appears naturally in Witten's M-theory construction of the model [39] and also in the context of the relation between $\mathcal{N}=2$ theories and integrable systems [40],

$$(t-\tilde{\Lambda}^N)\left(t-\prod_{i=1}^{N}(x+m_i)\right)=0\,. \tag{6.27}$$

The non-vanishing periods can then be written as integrals over the Seiberg-Witten differential. In terms of the variables x and t, the periods are

$$\mathcal{P}_{kl}=\int_{e_k}^{e_l} d\lambda_{\rm SW}=\int_{e_k}^{e_l} x\frac{dt}{t}\,, \tag{6.28}$$

which gives

$$\begin{aligned}\mathcal{P}_{kl}&=\sum_{i=1}^{N}\int_{e_k}^{e_l}\frac{x}{x+m_i}\\ &=N(e_l-e_k)-\sum_{i=1}^{N}m_i\log\left(\frac{e_l+m_i}{e_k+m_i}\right)\,.\end{aligned} \tag{6.29}$$

Thus, up to an overall numerical factor, the period $\mathcal{P}_{kl}$ is equal to the central charge Z_{kl} given in (6.15) as advertised above.

Several features of this correspondence require further comment. First, the twisted masses of the two-dimensional theory are now identified as hypermultiplet masses μ_i in the four-dimensional theory. Even in the weak-coupling limit, this seems to disagree with the classical correspondence, where the twisted masses were identified instead with the adjoint scalar VEVs a_i. In fact, there is actually no disagreement because, at least in the weak coupling regime, we have $a_i=\mu_i$ at the singular point. Second, as in the classical case, only the massive BPS states of the four-dimensional theory have counterparts in the two dimensional theory. For $|\mu_i-\mu_j|>>\Lambda$, the curve (6.25) describes a weakly-coupled four-dimensional theory with N massless quarks and an infinite number of stable BPS states. For $|\mu_i-\mu_j|<<\Lambda$ the curve describes a strongly-coupled theory in four-dimensions with N massless dyons. At least in the simplest case $N=2$, which corresponds to gauge group $SU(2)$ in four dimensions, it is known that the strongly-coupled theory has only a finite number of stable BPS states [6, 19, 20]. This is possible because the two regions of parameter/moduli space in the four-dimensional theory are disconnected by a surface of marginal stability of real codimension one. The explicit form of this surface has recently been determined in [20]. In the above, we found that a curve of marginal stability in the complex mass plane was required in order to produce a consistent description of the BPS spectrum of the two-dimensional theory with $N=2$. In this case, the existence of the required CMS can be deduced from the analysis of the corresponding four-dimensional theory given in [20].

JHEP11(1998)005

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A. The Lagrangian in component fields

In section 2, we considered the most general supersymmetric action with up to two derivatives or four-fermions for chiral superfields Φ_i,

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_F = \int d^4\theta\, K(\Phi_i, \bar{\Phi}_i) \,+ \int d^2\theta\, W(\Phi_i) \,+ \int d^2\bar{\theta}\, \bar{W}(\bar{\Phi}_i). \tag{A.1}$$

Minimal coupling of the each chiral multiplet to an abelian gauge superfield V (with equal charges) is generated by the replacement.

$$\Phi_i \to \exp(V)\Phi_i. \tag{A.2}$$

This Appendix provides a formula for the resulting Lagrangian in terms of component fields. In particular, explicit results will be given for the component Lagrangian of the CP^1 σ-model with twisted mass terms.

As in section 1, we will derive the two-dimensional component Lagrangian by dimensional reduction from a theory in four-dimensions with $\mathcal{N} = 1$ supersymmetry. The component expansion of the *four-dimensional* Lagrangian defined by $\mathcal{L}_D$ with the minimal coupling prescription (A.2) can be read off from formula (24) of [41]. Rewriting this formulae in the notation of section 1 yields

$$\begin{aligned}\mathcal{L}_D = {} & g_{i\bar{j}}\left(-D_m\phi^i D^m\bar{\phi}^j - \bar{\psi}^j_{\dot{\alpha}}\bar{\sigma}^{m\dot{\alpha}\alpha}\mathcal{D}_m\psi^i_\alpha + F^i\bar{F}^j\right) - \\ & -\frac{D}{2}\left(\phi^i\frac{\partial K}{\partial\phi^i} + \bar{\phi}_i\frac{\partial K}{\partial\bar{\phi}_i}\right) + ig_{i\bar{j}}\left(\bar{\phi}^j\psi^{i\alpha}\chi_\alpha + \phi^i\bar{\psi}^j_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}\right) - \\ & -\frac{1}{2}F^i g_{i\bar{l}}\Gamma^{\bar{l}}_{\bar{j}\bar{k}}\bar{\psi}^j_{\dot{\alpha}}\bar{\psi}^{k\dot{\alpha}} - \frac{1}{2}\bar{F}^i g_{l\bar{i}}\Gamma^l_{jk}\psi^{j\alpha}\psi^k_\alpha + \frac{1}{4}g_{i\bar{j},k\bar{l}}\psi^{i\alpha}\psi^k_\alpha\bar{\psi}^j_{\dot{\alpha}}\bar{\psi}^{l\dot{\alpha}}\end{aligned} \tag{A.3}$$

with the usual definitions for the tensors which arise naturally on a Kähler manifold.

$$\begin{aligned} g_{i\bar{j}} &= \frac{\partial^2 K}{\partial\phi^i\partial\bar{\phi}^j}\,, \\ g_{i\bar{j},k} &= \frac{\partial}{\partial\phi^k}\,g_{i\bar{j}} = g_{l\bar{j}}\Gamma^l_{ik}\,, \\ g_{i\bar{j},\bar{k}} &= \frac{\partial}{\partial\bar{\phi}^k}\,g_{i\bar{j}} = g_{i\bar{l}}\Gamma^{\bar{l}}_{\bar{j}\bar{k}}\,, \\ g_{i\bar{j},k\bar{l}} &= \frac{\partial^2}{\partial\phi^k\partial\bar{\phi}^l}\,g_{i\bar{j}}\,. \end{aligned} \tag{A.4}$$

Here D_m denotes the gauge-covariant derivative for a $U(1)$ gauge field V_m: $D_m = \partial_m + iV_m$. The fermion kinetic term in (A.3) contains a derivative which is also covariant with respect to general coordinate transformations,

$$\mathcal{D}_m \psi^i_\alpha = D_m \psi^i_\alpha + \Gamma^i_{jk}(D_m \phi^j)\psi^k_\alpha \,. \tag{A.5}$$

The F-term Lagrangian is not modified by minimal coupling and is given by

$$\mathcal{L}_F = F^i \frac{\partial W}{\partial \phi^i} + \bar{F}^i \frac{\partial \bar{W}}{\partial \bar{\phi}^i} - \frac{1}{2}\frac{\partial^2 W}{\partial \phi^i \partial \phi^j}\psi^{i\alpha}\psi^j_\alpha - \frac{1}{2}\frac{\partial^2 \bar{W}}{\partial \bar{\phi}^i \partial \bar{\phi}^j}\bar{\psi}^i_{\dot{\alpha}}\bar{\psi}^{j\dot{\alpha}} \,. \tag{A.6}$$

As in section 1, the two-dimensional fields in each chiral multiplet are a complex scalar ϕ^i, a Dirac fermion with components $\psi^i_- = \psi^i_1$ and $\psi^i_+ = \psi^i_2$ as well as a complex auxiliary field F^i. The gauge multiplet fields include the complex scalar $\sigma = V_1 - iV_2$, and the 2D gauge field with components $v_0 = V_0$ and $v_1 = V_3$ as well as a fermion with components χ_α and a real auxiliary field D. The two-dimensional Lagrangian is then obtained from (A.3) by dimensional reduction in the X_1 and X_2 directions.

In sections 3, 4 and 5, the case of a $U(1)$ gauge theory with two chiral multiplets is analysed in detail. In this case the low-energy effective theory is a variant of the supersymmetric CP^1 σ-model which includes twisted mass terms. The Lagrangian is a D-term for a single chiral superfield[8] W with components (w, ψ_α, F) and minimal coupling to a background gauge superfield $\hat{V}$ with expectation value,

$$\langle \hat{V} \rangle = -\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \hat{V}_m \,, \tag{A.7}$$

where $\hat{V}_1 = \mathrm{Re}(m)$, $\hat{V}_2 = -\mathrm{Im}(m)$ and $\hat{V}_0 = \hat{V}_3 = 0$. Note that the fermion and auxiliary field components of $\hat{V}$ are zero. The superspace Lagrangian is obtained by setting $N = 2$ in (3.9),

$$\mathcal{L} = r\int d^4\theta \, \log\left(1 + \bar{W}\exp\left(2\langle \hat{V}\rangle\right) W\right). \tag{A.8}$$

The component Lagrangian follows from Equations (A.3) and (A.4) with $\Phi_1 = W$ $V = \langle \hat{V} \rangle$, $K(W, \bar{W}) = r\log(1 + \bar{W}W)$. The final result is

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} \\
\mathcal{L}^{(0)} &= -\frac{1}{\rho^2}\left[r\left(\partial_\mu \bar{w}\partial^\mu w + |m|^2|w|^2\right) + \frac{\theta}{2\pi}\varepsilon^{\mu\nu}\partial_\mu \bar{w}\partial_\nu w \right] \\
\mathcal{L}^{(2)} &= \frac{1}{\rho^2}\left[i\bar{\psi}_-\left(\partial_+ - \frac{2i}{\rho}\mathrm{Im}(\bar{w}\partial_+ w)\right)\psi_- + i\bar{\psi}_+\left(\partial_- - \frac{2i}{\rho}\mathrm{Im}(\bar{w}\partial_- w)\right)\psi_+ \right] \\
&\qquad\qquad - \frac{i}{\rho^2}\left(1 - \frac{2|w|^2}{\rho}\right)\left(m\bar{\psi}_-\psi_+ - \bar{m}\bar{\psi}_-\psi_+\right) \\
\mathcal{L}^{(4)} &= \frac{r}{\rho^2}\left[\left(F - \frac{2\bar{w}\psi_-\psi_+}{\rho}\right)\left(\bar{F} - \frac{2w\bar{\psi}_-\bar{\psi}_+}{\rho}\right) + \frac{2}{\rho^2}\psi_-\psi_+\bar{\psi}_-\bar{\psi}_+\right],
\end{aligned} \tag{A.9}$$

where $\rho = 1 + |w|^2$ and $\partial_\pm = \partial_0 \pm \partial_1$.

[8]Hopefully denoting the chiral superfield with the same letter as the superpotential will not cause confusion as the latter is zero in this example.

JHEP11(1998)005

B. Small fluctuation analysis

This Appendix provides further details of the small-fluctuation expansion around the soliton and dyon background. Specifically, we will start from the component Lagrangian (A.9) derived in the previous appendix and expand the fields w, ψ_α and F around their values in the general time-dependent dyon solution,

$$w_{\rm cl} = \exp\left(\sqrt{|m|^2 - \omega^2}\, x \,+\, i\omega t\right) \tag{B.1}$$

and $\psi^\alpha_{\rm cl} = F_{\rm cl} = 0$. In this way we will derive the formulae (5.8)–(5.11) which were used in the calculation of quantum corrections to the soliton mass in section 5. In the process we will also derive (5.24) which was the starting point for the calculation of the weak-coupling monodromy of the dyon spectrum given in the text. We consider the expansion of the bose and fermi fields in turn.

B.1. Bosons

In section 4, the bosonic Lagrangian $\mathcal{L}^{(0)}$ is simplified by the change of variables,

$$w = \tan\left(\frac{\varphi}{2}\right)\exp(i\alpha)\,. \tag{B.2}$$

The Lagrangian is given in terms of φ and α in equation (4.7) of section 4. The classical values of these fields are $\varphi_{\rm cl} = \varphi_\omega(x)$ and $\alpha_{\rm cl} = \omega t$ where $\varphi_\omega(x)$ is defined in equation (4.13). To determine the Lagrangian for the quadratic fluctuations of the bosonic fields, it is useful to perform yet another change of variables and write the Lagrangian in terms of a three component unit vector $\hat{\mathbf{n}} = (n_1, n_2, n_3)$, $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = 1$. The scalar field w is given in terms of the components of $\hat{\mathbf{n}}$ by

$$w = \frac{n_1 + in_2}{1 - n_3}\,. \tag{B.3}$$

In addition we may express $\mathbf{n}$ in terms of φ and α as

$$\mathbf{n} = (\sin\varphi\cos\alpha,\ \sin\varphi\sin\alpha,\ -\cos\varphi)\,. \tag{B.4}$$

With this change of variables, (3.10) becomes (for $\theta = 0$)

$$\mathcal{L}^{(0)} = -\frac{r}{4}\left[\partial_\mu\hat{\mathbf{n}}\cdot\partial^\mu\hat{\mathbf{n}} + |m|^2(1 - n_3^2)\right]. \tag{B.5}$$

For $m = 0$, this is the action of the $O(3)$ σ-model and the change of variables reflects the standard equivalence between σ-models with target space CP^1 and $O(3)$. The effect of a non-zero twisted mass m is to introduce a potential on the target manifold which has minima at the north and south poles $n_3 = \pm 1$. These points are the two supersymmetric vacua of the theory.

Next we expand $\mathbf{n}$ around its classical value

$$\mathbf{n}_{\rm cl} = (\sin(\varphi_\omega)\cos(\omega t), \sin(\varphi_\omega)\sin(\omega t), -\cos(\varphi_\omega))\,. \tag{B.6}$$

It is convenient to parametrize the fluctuations of the constrained field $\mathbf{n}$ in terms of two unconstrained real variables u_1 and u_2,

$$\begin{aligned}\mathbf{n} &= \mathbf{n}_{\rm cl} + \delta\mathbf{n} \\ &= (\sin(\varphi_\omega{+}u_2)\cos(\omega t){-}u_1\sin(\omega t), \sin(\varphi_\omega{+}u_2)\sin(\omega t){+}u_1\cos(\omega t), -\cos(\varphi_\omega{+}u_2))\,.\end{aligned} \tag{B.7}$$

Expanding to quadratic order in bosonic fluctuations we have,

$$\mathcal{L}^{(0)} = \mathcal{L}^{(0)}[\varphi_{\rm cl}, \alpha_{\rm cl}] - \frac{r}{4}\vec{u}^{\rm T}\cdot\mathcal{M}_{\rm B}\cdot\vec{u} + O(\vec{u}^3)\,, \tag{B.8}$$

where $\vec{u}^{\rm T} = (u_1, u_2)$ and

$$\mathcal{M}_{\rm B} = \begin{pmatrix} \frac{\partial^2}{\partial t^2} + \Delta_{\rm B}(\omega) & 2\omega\cos(\varphi_\omega)\frac{\partial}{\partial t} \\ -2\omega\cos(\varphi_\omega)\frac{\partial}{\partial t} & \frac{\partial^2}{\partial t^2} + \Delta_{\rm B}(\omega) \end{pmatrix}, \tag{B.9}$$

with

$$\Delta_{\rm B}(\omega) = -\frac{\partial^2}{\partial x^2} + (|m|^2 - \omega^2)\cos(2\varphi_\omega)\,. \tag{B.10}$$

Setting $\omega = 0$, (B.9) reproduces the bosonic terms in (5.8)–(5.11)

B.2. Fermions

The terms in the expansion which are bilinear in the fermion fields ψ_α and $\bar{\psi}_{\dot\alpha}$ and independent of the bosonic fluctuations may be calculated by replacing w by its classical value $w_{\rm cl}$ in the fermion bilinear Lagrangian $\mathcal{L}^{(2)}$ which appears in (A.9),

$$\begin{aligned}\mathcal{L}^{(2)}[\psi, \bar\psi, w_{cl}] = \frac{1}{\rho_{\rm cl}^2}\Big[& i\bar\psi_-\left(\partial_+ - \frac{2i}{\rho_{\rm cl}}{\rm Im}(\bar{w}_{\rm cl}\partial_+ w_{\rm cl})\right)\psi_- + \\ & + i\bar\psi_+\left(\partial_- - \frac{2i}{\rho_{\rm cl}}{\rm Im}(\bar{w}_{\rm cl}\partial_- w_{\rm cl})\right)\psi_+\Big] \\ & - \frac{i}{\rho_{\rm cl}^2}\left(1 - \frac{2|w_{\rm cl}|^2}{\rho_{\rm cl}}\right)\left(m\bar\psi_-\psi_+ - \bar{m}\bar\psi_-\psi_+\right),\end{aligned} \tag{B.11}$$

where $\rho_{\rm cl} = 1 + |w_{\rm cl}|^2$. Introducing the γ-matrices $\gamma^0 = i\sigma_2$, $\gamma^1 = -\sigma_1$ and $\gamma^5 = \gamma^0\gamma^1 = -\sigma_3$ and Dirac fermions;

$$\tilde\Psi = \begin{pmatrix}\psi_- \\ \psi_+\end{pmatrix}, \qquad \tilde\Psi^\dagger = \begin{pmatrix}\bar\psi_- \\ \bar\psi_+\end{pmatrix}. \tag{B.12}$$

Defining $\Psi = \exp(i\pi\gamma^5/4)\tilde\Psi$ and $\bar\Psi = \Psi^\dagger\gamma^0$ we get

$$\mathcal{L}^{(2)}[\psi, \bar\psi, w_{\rm cl}] = -\frac{ir}{4}\bar\Psi_\alpha\hat{\mathcal{M}}_{\rm F}^{\alpha\beta}\Psi_\beta\,, \tag{B.13}$$

with

$$\hat{\mathcal{M}}_{\rm F} = \gamma^\mu \left(\partial_\mu + \frac{2i}{\rho_{\rm cl}} {\rm Im}(\bar{w}_{\rm cl}\partial_\mu w_{\rm cl})\right) + \left(1 - \frac{2|w_{\rm cl}|^2}{\rho_{\rm cl}^2}\right) \left[{\rm Re}(m) I + i{\rm Im}(m)\gamma^5\right] \Psi \,. \quad \text{(B.14)}$$

Substituting for $w_{\rm cl}$ as in (B.1) yields

$$\hat{\mathcal{M}}_{\rm F} = \gamma^\mu \partial_\mu + \cos\left(\varphi_\omega(x)\right) \left[i\omega\gamma^0 + {\rm Re}(m) I + i{\rm Im}(m)\gamma^5\right] . \quad \text{(B.15)}$$

This is equation (5.24) of section 5. In the special case of the static soliton $\omega = 0$, we may rewrite (B.14) in a form which makes the supersymmetry between boson and fermion fluctuations manifest with the following three steps:

1: In the special case of the static soliton with $\omega = 0$, the 2D chiral anomaly for the Dirac operator (B.14), which is derived in section 5, vanishes. In this case only, we may absorb all dependence on the phase of m by a chiral redefinition of the Dirac spinors,

$$\Psi \to \Psi' = \exp\left(i\frac{\beta}{2}\gamma^5\right) \Psi \,, \qquad \bar{\Psi} \to \bar{\Psi}' = \bar{\Psi} \exp\left(i\frac{\beta}{2}\gamma^5\right) , \quad \text{(B.16)}$$

where $\beta = \arg(m)$.

2: It is also convenient to choose a new basis for the spinors Ψ and $\bar{\Psi}$ in which the γ-matrices become $\gamma^0 = i\sigma_2$, $\gamma^1 = -\sigma_3$ and $\gamma^5 = \gamma^0\gamma^1 = \sigma_1$. This is acomplished by the following unitary tranformation

$$\Psi' \to \Psi'' = \exp\left(i\frac{\pi}{4}\sigma_2\right) \Psi' \,, \qquad \Psi'^\dagger \to \Psi''^\dagger = \Psi'^\dagger \exp\left(-i\frac{\pi}{4}\sigma_2\right) . \quad \text{(B.17)}$$

3: We define two real Majorana fermions $\rho_{1\alpha}$ and $\rho_{2\alpha}$, with $\rho_i^\dagger = \rho_i^{\rm T}$, which are related to the Dirac fermion Ψ'' by $\Psi'' = \rho_1 + i\rho_2$.

Finally, the bilinear action for the Majorana fermions $\vec{\rho}_\alpha^{\,\rm T} = (\rho_{1\alpha}, \rho_{2\alpha})$ is

$$\mathcal{L}^{(2)}[\psi, \bar{\psi}, w_{\rm cl}] = -\frac{r}{4} \vec{\rho}_\alpha^{\,\rm T} \cdot \mathcal{M}_{\rm F}^{\alpha\beta} \cdot \vec{\rho}_\beta \,, \quad \text{(B.18)}$$

where

$$\mathcal{M}_{\rm F}^{ij} = \delta^{ij} \begin{pmatrix} \frac{\partial}{\partial t} & D \\ -D^T & \frac{\partial}{\partial t} \end{pmatrix} \quad \text{(B.19)}$$

and

$$D = \frac{\partial}{\partial x} + |m| \cos\left(\varphi_S(x)\right) , \qquad D^T = -\frac{\partial}{\partial x} + |m| \cos\left(\varphi_S(x)\right) . \quad \text{(B.20)}$$

This is gives the fermionic part of eqs. (5.8)–(5.11).

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